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Examples on use of vector analysis in physics

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1. Vector functions, fields

1.1 A moving particle

Position of a particle in space is determined by the position vector \( \mathbf{r} = \mathbf{r}(t) \). The particle velocity and acceleration are specified by the formulas

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}.
\]  

(1.1)

1.2 Fields in fluid mechanics. Streamlines.

Fluid motion is usually described with the help of the scalar fields of pressure \( P = P(\mathbf{r}, t) \), density \( \rho = \rho(\mathbf{r}, t) \), temperature \( T = T(\mathbf{r}, t) \), and a vector field of the flow velocity \( \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \). The velocity field may be depicted by use of streamlines. By definition, a tangent unit vector to a streamline \( \hat{\tau} \) points in the direction of velocity \( \mathbf{u} \).

Calculating a unit tangent vector as

\[
\hat{\tau} = \frac{d\mathbf{r}}{dr}, \quad \text{with} \quad d\mathbf{r} = (dx, dy, dz)
\]

we find

\[
\hat{\tau} = \frac{d\mathbf{r}}{dr} = \frac{\mathbf{u}}{u},
\]

that is

\[
\frac{dx}{dr} = \frac{u_x}{u}, \quad \frac{dy}{dr} = \frac{u_y}{u}, \quad \frac{dz}{dr} = \frac{u_z}{u}
\]

or

\[
\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}.
\]

(1.2)

In the same way we can calculate field-lines for other vector fields.

**Example:** Find stream-lines of the flow \((u_x, u_y) = (\Omega y, -\Omega x)\).

We have to solve the equation

\[
\frac{dx}{\Omega y} = -\frac{dy}{\Omega x}, \quad \text{or} \quad xdx + ydy = 0.
\]

We find
\[ x^2 + y^2 = \text{const}. \]
The streamlines are circles.

1.3 Fields in electrodynamics
In electrodynamics we work with the scalar fields of potential \( \phi = \phi(\mathbf{r},t) \), and charge density \( \rho_e = \rho_e(\mathbf{r},t) \). The most important vector fields used in electrodynamics are the electric field \( \mathbf{E} = \mathbf{E}(\mathbf{r},t) \), the magnetic field \( \mathbf{B} = \mathbf{B}(\mathbf{r},t) \) and the current density \( \mathbf{j} = \mathbf{j}(\mathbf{r},t) \).

2. Derivatives and integrals of vector functions
2.1 Electric force on distributed charge
Electric charge distributed in space is described with the help of charge density
\[
\rho_e = \frac{dq}{dV},
\]
which determines elementary charge in the position \( \mathbf{r} \) as
\[
dq = \rho_e(\mathbf{r})dV = \rho_e \, dx \, dy \, dz.
\]
**Example:** The charge is placed in the electric field \( \mathbf{E} = \mathbf{E}(\mathbf{r}) \). What is the total force acting on the charge inside some volume \( V_0 \)?

**Solution:**
The force acting on the elementary charge is
\[
d\mathbf{F} = \mathbf{E} \, dq = \mathbf{E} \rho_e \, dV.
\]
Then the total force is calculated as
\[
\mathbf{F} = \int d\mathbf{F} = \int \mathbf{E} \, dq = \iiint_{V_0} \mathbf{E} \rho_e \, dV.
\]

2.2 Gravitational force on distributed mass
Mass distribution inside a star with the gravitational field \( \mathbf{g} = \mathbf{g}(\mathbf{r}) \) is described by the density
\[
\rho = \frac{dM}{dV}.
\]
The gravitational field is, actually, the gravitational acceleration. What is the total force acting on the mass inside some volume \( V_0 \)?

**Solution:**

The force acting on the elementary mass \( dM = \rho dV \) is

\[
\mathbf{dF} = \mathbf{g} dM = \mathbf{g} \rho \, dV .
\]

Then the total force is calculated as

\[
\mathbf{F} = \int d\mathbf{F} = \int \mathbf{g} \, dM = \iiint_{V_0} \mathbf{g} \rho \, dV .
\]  

(2.4)

(2.5)

3. Gradient

3.1 Particle motion in a potential field

A particle of mass \( m \) moves with velocity \( \mathbf{v} \) in a potential field with force \( \mathbf{F} = -\nabla \phi (\mathbf{r}) \).

Show that total particle energy \( \frac{1}{2} m \mathbf{v}^2 + \phi \) is a constant.

**Solution:**

Particle motion is described by Newton’s law

\[
m \frac{d\mathbf{v}}{dt} = \mathbf{F} = -\nabla \phi .
\]

Multiplying both parts of the equation by the particle velocity \( \mathbf{v} = \frac{d\mathbf{r}}{dt} \)

\[
m \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{r}}{dt} \cdot \mathbf{F} = -\frac{d\mathbf{r}}{dt} \cdot \nabla \phi
\]

we find

\[
\frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 \right) = -\frac{d\phi}{dt} ,
\]  

that is \[
\frac{d}{dt} \left( \frac{1}{2} m \mathbf{v}^2 + \phi \right) = 0
\]

or \[
\frac{1}{2} m \mathbf{v}^2 + \phi = \text{const} .
\]
4. Line integrals

4.1 Work of a force

A particle moving from a point P to a point Q along a curve L experiences a force \( \mathbf{F} \), which is not necessarily a potential one. What is the total work of the force?

**Solution:**

The elementary work is \( dW = \mathbf{F} \cdot d\mathbf{r} \). Then the total work is

\[
W = \int_P^Q dW = \int_L \mathbf{F} \cdot d\mathbf{r}.
\]

4.2 Work of a gravitational force

What is the total work of the gravitational force when we move the mass \( m \) along a line \( L \) ascending by the total height \( \Delta z \)? The gravity acceleration is constant \( g = (0, 0, -g) \).

**Solution:**

The gravitational force is \( \mathbf{F} = mg = -m\nabla \phi \), where the potential \( \phi = -\mathbf{g} \cdot \mathbf{r} \) or \( \phi = gz \).

Then the work of the gravitational force is

\[
W = \int_L \mathbf{F} \cdot d\mathbf{r} = -\int_L m\nabla \phi \cdot d\mathbf{r} = -m(\phi_Q - \phi_P) = -mg\Delta z.
\]

4.3 Electric force on charge distributed along a wire

Electric charge distributed along a wire is described with the help of the line charge density

\[
\rho_l = \frac{dq}{dl}.
\]

A charged wire is placed in an electric field \( \mathbf{E} = \mathbf{E}(\mathbf{r}) \). What is the total force acting on the charge? The shape of the wire is described by a curve \( L \).

**Solution:**

The force acting on the elementary charge is

\[
d\mathbf{F} = \mathbf{E} \, dq = \mathbf{E} \rho_l \, dl.
\]

Then the total force is calculated as

\[
\mathbf{F} = \int d\mathbf{F} = \int_L \mathbf{E} \, dq = \int_L \mathbf{E} \rho_l \, dl.
\]
4.4 Magnetic force on a wire with current

A wire with current \( I \) is placed in a magnetic field \( \mathbf{B} = \mathbf{B}(\mathbf{r}) \). What is the total force acting on the charge? The shape of the wire is described by a curve \( L \).

**Solution:**

The force acting on the current element \( I \mathbf{d} \mathbf{r} \) of the wire is determined by Ampere’s law

\[
d\mathbf{F} = I \mathbf{d} \mathbf{r} \times \mathbf{B},
\]

here and below we are working in the SI units. Then the total force is calculated as

\[
\mathbf{F} = \int d\mathbf{F} = I \int_{L} d\mathbf{r} \times \mathbf{B}.
\]  

(4.1)

5. Surface integrals

5.1 Volumetric flow rate (discharge) of a fluid.

We have a uniform flow of a fluid with velocity \( U_1 \) in a tube with cross-sectional area \( S_1 \), see Fig. 4.1. How much fluid comes into the tank per unit time?

**Solution:**

The amount of the fluid coming into the tank is determined by the volume of the cylinder with the tube cross-section at the bottom, \( S_1 \), and the height \( U_1 \, dt \), hence

\[
dV = S_1 U_1 \, dt,
\]

Fig. 4.1. Fluid comes into the tank and leaves the tank: illustration of the discharge.
which determines the volumetric flow rate

\[ \frac{dV}{dt} = S_0 U_1. \]

In the case of an arbitrary surface and an arbitrary flow \( \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \) we can divide the surface area into elements \( dS = \hat{n} dS \), where \( \hat{n} \) is the normal unit vector, see Fig. 4.2. Then the elementary fluid volume passing \( dS \) per unit time is determined by the local velocity of the flow \( \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \)

\[ (dV)_{ds} = (\mathbf{u} \, dt) \cdot dS \quad \text{or} \quad \left( \frac{dV}{dt} \right)_{ds} = \mathbf{u} \cdot dS. \quad (5.1) \]

The total discharge (the volume of the fluid passing through \( S_0 \) per unit time) is

\[ \frac{dV}{dt} = \iint_{S_0} \mathbf{u} \cdot dS. \quad (5.2) \]

If the surface \( S_0 \) is closed, then Eq. (5.2) specify the discharge of the fluid going out of the volume enveloped by \( S_0 \), since conventionally the normal unit vector \( \hat{n} \) points out. This holds even if locally the fluid flows into the volume, because in that case \( \mathbf{u} \cdot dS < 0 \).
5.2 Mass flux in fluid mechanics.
We have an arbitrary surface $S_0$ and an arbitrary flow $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$. What is the mass flux through $S_0$ (how much mass passes $S_0$ per unit time)?

Solution:
We cut $S_0$ into small surface elements $dS$. Fluid volume passing through a surface element $dS$ per unit time is determined by the elementary discharge Eq. (5.1). Then the elementary mass flux is determined by the volume $(dV)_{dS}$ and the fluid density $\rho$

$$(dM)_{dS} = \rho (dV)_{dS} = \rho \mathbf{u} \cdot dS,$$

(5.3)
or

$$\left(\frac{dM}{dt}\right)_{dS} = \rho \mathbf{u} \cdot dS,$$

(5.4)
and the total mass flux is

$$\frac{dM}{dt} = \int_S \rho \mathbf{u} \cdot dS.$$ 

(5.5)

Similar calculations may be performed for the momentum and energy fluxes.

5.3 Electric current.
How can we find the electric current through a surface $S_0$ if we know charge distribution $\rho_e = \rho_e(\mathbf{r},t)$ and the microscopic motion of charged particles $\mathbf{u} = \mathbf{u}(\mathbf{r},t)$.

Solution:
By definition, electric current $I$ through $S_0$ is charge passing the surface per unit time $I \equiv (dq / dt)_{S_0}$. We start with the simplest case of the current produced by particles of one type. Similar to the mass flux in fluid mechanics, the charge passing the elementary surface $dS$ per unit time is determined by the particles inside the elementary volume $(dV)_{dS}$, that is

$$(dq)_{dS} = \rho_e (dV)_{dS} = \rho_e \mathbf{u} dt \cdot dS,$$

(5.6)
and

$$\left(\frac{dI}{dt}\right)_{dS} = \rho_e \mathbf{u} \cdot dS.$$ 

(5.7)
Then we calculate the total current as

$$I = \frac{dq}{dt} = \int_{S_0} \rho_e u \cdot dS.$$  \hspace{1cm} (5.8)

The above calculations are used to define the current density \( \mathbf{j}(\mathbf{r}, t) \):

$$(dI)_{ds} = \mathbf{j} \cdot dS, \quad \text{or} \quad I = \int_{S_0} \mathbf{j} \cdot dS,$$  \hspace{1cm} (5.9)

which relates the current density to the microscopic parameters

$$\mathbf{j} = \rho_e \mathbf{u}.$$  \hspace{1cm} (5.10)

Now let us consider the general case of different types of particles contributing to the current, as it takes place in plasma or an electrolyte. In that case the charge density is determined by different particles

$$\rho_e = \frac{dq}{dV} = \frac{1}{dV} \sum dq_i = \sum \rho_{ei},$$  \hspace{1cm} (5.11)

and the elementary current is

$$(I)_{ds} = \mathbf{j} \cdot dS = \left(\frac{dq}{dt}\right)_{ds} = \sum \left(\frac{dq_i}{dt}\right)_{ds} = \sum \rho_{ei} \mathbf{u}_i \cdot dS,$$  \hspace{1cm} (5.12)

which reduces to the formula for the current density

$$\mathbf{j} = \sum \rho_{ei} \mathbf{u}_i.$$  \hspace{1cm} (5.13)

5.4 Heat flux due to thermal conduction.

Suppose that we have two adjacent gas volumes at equal uniform pressure and different temperatures, \( T_1 > T_2 \). If we put the volumes into contact, then after some time the gas temperature will be uniform and equal some intermediate value \( T^* \) with \( T_1 > T^* > T_2 \). The temperature variation imply that energy has been redistributed, though we had no hydrodynamic motion (pressure was constant and there was no force, which could make the gas move). In that case the energy was redistributed by use of random thermal motion of gas particles. Physical processes related to the random motion are called transport processes. The most important transport processes are thermal conduction, diffusion and viscosity. In this example we consider heat flux due to thermal conduction. How much
heat $H$ (thermal energy) passes through a surface $S_0$ because of thermal conduction if the temperature field is $T = T(r, t)$?

**Solution:**

First we consider heat flux through the elementary surface $dS$. In order to describe the heat flux we introduce the flux density similar to the current density Eq. (5.9) as

$$\left( \frac{dH}{dt} \right)_{ds} = \mathbf{j}_H \cdot dS.$$  \hspace{1cm} (5.14)

It has been obtained experimentally that the flux density is proportional to the temperature gradient

$$\mathbf{j}_H = -\kappa \nabla T;$$ \hspace{1cm} (5.15)

equation (5.15) is known as Fick’s law. The minus sign comes to Eq. (5.15) because heat is transferred from the regions of larger temperature to the cold regions. The factor $\kappa$ is called the coefficient of thermal conduction. Then the elementary heat flux is

$$\left( \frac{dH}{dt} \right)_{ds} = -\kappa \nabla T \cdot dS,$$ \hspace{1cm} (5.16)

and the total flux is

$$\frac{dH}{dt} = \iiint_{S_0} \mathbf{j}_H \cdot dS = -\iiint_{S_0} \kappa \nabla T \cdot dS.$$  \hspace{1cm} (5.17)

**5.5 Diffusion.**

Another example of a transport process is diffusion. Suppose that we have a large volume filled by a gas $a$ in mechanic equilibrium. Besides, somewhere in the volume we have a little amount of gas $b$, which does not disturb the equilibrium (for example, particles of smoke or perfume in the air). The amount of gas $b$ may be described by concentration $c = c(r, t)$, which is the number of particles $b$ in elementary volume

$$c = \frac{dN_b}{dV}.$$ \hspace{1cm} (5.18)

How can we describe flux of gas $b$ through a surface $S_0$ due to thermal motion?
**Solution:**

In order to describe flux through an elementary surface $d\mathbf{S}$ we introduce the flux density

$$\left( \frac{dN_b}{dt} \right)_{ds} = j_D \cdot d\mathbf{S}, \quad (5.19)$$

where the flux density is related to the concentration gradient according to Fick’s law

$$j_D = -D\nabla c. \quad (5.20)$$

The relation like (5.20) is typical for all transport processes. The factor $D$ in Eq. (5.20) is the diffusion coefficient. Then the elementary flux of gas $b$ is

$$\left( \frac{dN_b}{dt} \right)_{ds} = -D\nabla c \cdot d\mathbf{S}, \quad (5.21)$$

which results in the total flux

$$\frac{dN_b}{dt} = -\int_{S_0} D\nabla c \cdot d\mathbf{S}. \quad (5.22)$$

### 5.6 Pressure force in a fluid.

What is the pressure force produced by a fluid on a surface area $S_0$?

**Solution:**

By definition, pressure is a force per unit surface area directed normally to the surface. Choosing a surface element $d\mathbf{S}$ we have the elementary pressure force

$$d\mathbf{F} = -P d\mathbf{S}. \quad (5.23)$$

The minus sign comes to Eq. (5.23) because the normal unit vector to the surface is traditionally directed towards the fluid, but the pressure force acts in the opposite direction. Then the total pressure force is

$$\mathbf{F} = -\int_{S_0} P d\mathbf{S}. \quad (5.24)$$

Respectively, if we are interested in torque, then we calculate first the elementary torque

$$d\mathbf{T} = \mathbf{r} \times d\mathbf{F} = -\mathbf{r} \times (P d\mathbf{S})$$

and find the total torque by integration

$$\mathbf{T} = \int_{S_0} \mathbf{r} \times d\mathbf{F} = -\int_{S_0} P \mathbf{r} \times d\mathbf{S}. \quad (5.25)$$
5.7 Faraday's law in electrodynamics.

Faraday’s law is another important example on use of line and surface integrals. According to the Faraday’s law, time variation of a magnetic field generates rotational electric field. From the mathematical point of view it implies that circulation of electric field $E$ along any closed loop $C$ is proportional to time variations of the magnetic flux through any surface $S$ surrounded by the loop. The numerical coefficient in the law depends on the system of units; In the SI units we have

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$  \hspace{1cm} (5.26)


6.1 Incompressible flow.

We have an incompressible flow, for which fluid (gas) density does not change $\rho = \text{const}$. What is the differential equation for the velocity field $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$?

**Solution:**

We consider a closed arbitrary volume $V_0$ in the stream with neither sources nor sinks. Since the fluid density does not change, then amount of the fluid (measured by volume) coming into $V_0$ per unit time is equal to the amount of the fluid going out. In that case taking into account Eq. (5.2) we have

$$\frac{dV}{dt} = 0 = \iiint_{S_0} \mathbf{u} \cdot d\mathbf{S}.$$ \hspace{1cm} (6.1)

Using the Gauss theorem we transform Eq. (6.1) into a volume integral

$$0 = \iiint_{S_0} \mathbf{u} \cdot d\mathbf{S} = \iiint_{V_0} \nabla \cdot \mathbf{u} dV.$$ \hspace{1cm} (6.2)

Since the volume $V_0$ is arbitrary, then

$$\nabla \cdot \mathbf{u} = 0.$$ \hspace{1cm} (6.3)
6.2 Coulomb’s law in electrostatics.

We have two point charges \( Q \) and \( q \) separated by the distance \( r \). According to Coulomb’s law the force acting on \( q \) is

\[
F = \frac{qQ}{4\pi\varepsilon_0 r^2} \hat{r},
\]

(6.4)

where \( \hat{r} \) is the unit vector pointing from \( Q \) to \( q \), and \( \varepsilon_0 \) is a universal physical constant introduced in the SI. The electric field is defined in electrostatics as the force acting on unit charge

\[
E = \frac{F}{q} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}.
\]

(6.5)

Let us calculate the electric flux out of a sphere \( S_r \) with the centre in \( Q \) and radius \( r \).

The surface element of such a sphere is \( dS = \hat{n} dS = \hat{r} dS \). Then we have

\[
\oiint_{S_r} E \cdot dS = \oiint_{S_r} \frac{Q}{4\pi\varepsilon_0 r^2} \cdot dS = \frac{Q}{4\pi\varepsilon_0 r^2} \oiint_{S_r} dS = \frac{Q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{Q}{\varepsilon_0}.
\]

(6.6)

We can formulate the same result in a more general way for an arbitrary closed surface \( S_0 \) (this result is proved rigorously in vector analysis)

\[
\oiint_{S_0} E \cdot dS = \oint_{V_0} \frac{\rho_e}{\varepsilon_0} dV.
\]

(6.7)

The result Eq. (6.7) may be also presented in a differential form. With the help of the Gauss law we transform (6.7) as

\[
\oiint_{S_0} E \cdot dS = \iiint_{V_0} \nabla \cdot E dV = \iiint_{V_0} \frac{\rho_e}{\varepsilon_0} dV.
\]

(6.8)

Since the volume \( V_0 \) is arbitrary, then

\[
\nabla \cdot E = \frac{\rho_e}{\varepsilon_0}.
\]

(6.9)

Equation (6.9) is known as the Poisson equation, and it is considered as one of the basic results in electrodynamics.

A similar equation may be written for the magnetic field taking into account that there are no magnetic charges

\[
\nabla \cdot B = 0.
\]

(6.10)
6.3 The Poisson equation for the gravitational force.

Newton’s gravitational law has a form similar to Coulomb’s law in electrostatics, Eq. (6.4)

\[ F = -\gamma \frac{mM}{r^2} \hat{e}_r, \]  

(6.11)

where \( m \) and \( M \) are two point masses separated by the distance \( r \), and \( \gamma \) is the gravitational constant. We have the minus sign in Eq. (6.11) because two masses attract each other, while two positive charges repulse. The gravity acceleration plays the role of intensity of the gravitational field

\[ g = \frac{F}{m} = -\gamma \frac{M}{r^2} \hat{e}_r. \]  

(6.12)

Calculating flux of \( g \) out of a sphere with a centre in \( M \), we obtain

\[ \oint_{S_r} g \cdot dS = -\oint_{S_r} \frac{\gamma M}{r^2} \cdot dS = -\frac{\gamma M}{r^2} \oint_{S_r} dS = -\frac{\gamma M}{r^2} 4\pi r^2 = -4\pi \gamma M. \]  

(6.13)

In general, the same result may be presented as

\[ \oint_{S_0} g \cdot dS = -4\pi \gamma M_{\text{inside } S_0} = -4\pi \gamma \int_{V_0} \rho dV. \]  

(6.14)

Using the Gauss theorem we can present the same law in the differential form

\[ \nabla \cdot g = -4\pi \gamma \rho. \]  

(6.15)

6.4 Incompressible flow with sources.

The equation (6.3) for an incompressible flow changes, if we have sources of the fluid (gas), for example, due to chemical reactions or because of pumps. Suppose that the sources are distributed in an arbitrary volume \( V_0 \) with the density \( W \) and produce new fluid volume at the volumetric rate (discharge) \( Q = dV/dt \) so that

\[ Q = \iiint_{V_0} W dV. \]  

(6.16)

Then the amount of fluid leaving \( V_0 \) is equal

\[ \frac{dV}{dt} = Q = \iiint_{S_0} \mathbf{u} \cdot dS = \iiint_{V_0} \nabla \cdot \mathbf{u} dV, \]  

(6.17)

where we have taken into account Eq. (5.2). Comparing (6.16), (6.17) we obtain
\[ Q = \iiint_{V_0} \nabla \cdot \mathbf{u} \, dV = \iiint_{V_0} W \, dV. \]

Since the volume \( V_0 \) is arbitrary, then the velocity field obeys the equation
\[ \nabla \cdot \mathbf{u} = W. \tag{6.18} \]

6.5 Mass conservation in fluid mechanics.
We consider a flow described by the fields \( \mathbf{u} = \mathbf{u}(\mathbf{r}, t), \rho = \rho(\mathbf{r}, t) \) and an arbitrary fixed volume \( V_0 \) like that shown in Fig. 4.2. Mass conservation for the chosen volume may be described by an equation
\[
\text{(mass variation inside the volume)} = \\
- \text{(mass flux out of the volume)} + \text{(sources)}.
\tag{6.19}
\]

Total mass inside \( V_0 \) is
\[ M = \iiint_{V_0} \rho \, dV, \tag{6.20} \]
which varies in time as
\[ \frac{dM}{dt} = \frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \iiint_{V_0} \rho \, dV = \iiint_{V_0} \frac{\partial \rho}{\partial t} \, dV, \tag{6.21} \]
since the volume is fixed. We calculate mass flux out of \( V_0 \) according to Eq. (5.5) and use the Gauss theorem
\[
\left( \frac{dM}{dt} \right)_{\text{due to flux}} = \iiint_{S_0} \rho \mathbf{u} \cdot d\mathbf{S} = \iiint_{V_0} \nabla \cdot (\rho \mathbf{u}) \, dV. \tag{6.22}
\]
Sources of mass (for example, pumps) with distributed source density \( W_M(\mathbf{r}, t) \) produce
\[
\left( \frac{dM}{dt} \right)_{\text{due to sources}} = \iiint_{V_0} W_M \, dV. \tag{6.23}
\]
Substituting (6.21) – (6.23) into (6.19) we find
\[
\iiint_{V_0} \frac{\partial \rho}{\partial t} \, dV = - \iiint_{V_0} \nabla \cdot (\rho \mathbf{u}) \, dV + \iiint_{V_0} W_M \, dV. \tag{6.24}
\]
Since the volume \( V_0 \) is arbitrary, then we obtain the differential equation of mass conservation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = W_M .
\] (6.25)

If there are no sources (which is, actually, the most typical case), then Eq. (6.25) takes the form
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 .
\] (6.26)

Equation (6.26) is known as the continuity equation. If the fluid is incompressible \( \rho = \text{const} \), then the continuity equation reduces to \( \nabla \cdot \mathbf{u} = 0 \), that is to Eq. (6.3).

6.6 Conservation of electrical charge.

Similar to the previous example, we consider the fields of charge density \( \rho_e = \rho_e (\mathbf{r}, t) \), current density \( \mathbf{j} = \mathbf{j}(\mathbf{r}, t) \), and an arbitrary fixed volume \( V_0 \) like that shown in Fig. 4.2.

Charge conservation for the chosen volume may be described by an equation

\[
\frac{dq}{dt} = \int \int \int_{V_0} \rho_e dV - \int \int \int_{V_0} \mathbf{j} \cdot dS .
\] (6.27)

Charge variation inside \( V_0 \) is

\[
\frac{dq}{dt} = \frac{\partial q}{\partial t} = \frac{\partial}{\partial t} \int \int \int_{V_0} \rho_e dV = \int \int \int_{V_0} \partial_t \rho_e dV ,
\] (6.28)

Charge flux out of the volume is determined by the electric current according to Eq. (5.8)

\[
\left( \frac{dq}{dt} \right)_{\text{due to flux}} = I_{\text{out}} = \int \int \int_{S_{V_0}} \mathbf{j} \cdot dS = \int \int \int_{V_0} \nabla \cdot \mathbf{j} dV .
\] (6.29)

Then we find from Eq. (6.27)

\[
\int \int \int_{V_0} \partial_t \rho_e dV = - \int \int \int_{V_0} \nabla \cdot \mathbf{j} dV ,
\]

which may be reduced to the differential equation of charge conservation

\[
\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j} = 0 .
\] (6.30)

From the microscopic point of view in the case of one type of particles we have \( \mathbf{j} = \rho_e \mathbf{u} \) and the equation of charge conservation may be written in the same form as the continuity equation.
\[ \frac{\partial \rho_c}{\partial t} + \nabla \cdot (\rho_c u) = 0. \] (6.31)

6.7 Collective drift of particles.

In this example we consider conservation of the number of particles described by concentration, Eq. (5.18). Problems of this type are typical for combustion, where we have to take into account concentration of fuel, oxidizer, final and intermediate products of the reaction. In plasma physics and space science we also deal with a large number of different particles: electrons, ions of different type, neutrals, positrons, etc. As before, we consider a fixed volume \( V_0 \) shown in Fig. 4.2. Still, in the present example we may have particle flux caused by two different mechanisms: hydrodynamic flow and diffusion. We start with the hydrodynamic flow with no diffusion.

a) Drift due to a flow, no diffusion

The total number of particles \( b \) in \( V_0 \) is

\[ N_b = \iiint_{V_0} c \, dV, \] (6.32)

which varies in time as

\[ \frac{\partial N_b}{\partial t} = \frac{\partial}{\partial t} \iiint_{V_0} c \, dV = \iiint_{V_0} \frac{\partial c}{\partial t} \, dV. \] (6.33)

The continuity equation for the particle drift is

\[ \text{(variation of } N_b \text{ inside the volume)} = \]

\[ - \text{(flux of } N_b \text{ out of the volume)} + \text{(sources)}. \] (6.34)

Neglecting diffusion we have the flux due to the hydrodynamic flow only, which may be calculated similar to the mass flux, Eqs. (5.3) – (5.5)

\[ \left( \frac{dN_b}{dt} \right)_{\text{due to flux}} = \iiint_{S_0} (cu) \cdot dS = \iiint_{V_0} \nabla \cdot (cu) \, dV. \] (6.35)

When we consider continuity of different particles, then sources and sinks are quite typical. The number of particles of one type may change because of chemical reactions, ionization, recombination, birth and annihilation of electron-positron pairs, etc. Designating the density of sources/sinks by \( W_N \), we find
\[
\left( \frac{dN_b}{dt} \right)_{\text{due to sources}} = \iiint_{V_0} W_N \, dV .
\] (6.36)

Using Eq. (6.34) we obtain the continuity equation for the number of particles
\[
\iiint_{V_0} \frac{\partial c}{\partial t} \, dV = - \iiint_{V_0} \nabla \cdot (cu) \, dV + \iiint_{V_0} W_N \, dV ,
\] (6.37)
or
\[
\frac{\partial c}{\partial t} + \nabla \cdot (cu) = W_N .
\] (6.38)

The continuity equation holds for any type of particles.

b) Drift due to diffusion, no flow

In that case the flux out of \( V_0 \) in Eq. (6.34) is determined by the diffusion only, that is
\[
\left( \frac{dN_b}{dt} \right)_{\text{due to flux}} = - \int_{S_0} D \nabla c \cdot dS = - \iiint_{V_0} \nabla \cdot (D \nabla c) \, dV ,
\] (6.39)
see Eq. (5.22). Substituting Eqs. (6.33), (6.36), (6.39) into Eq. (6.34), we find the diffusion equation
\[
\iiint_{V_0} \frac{\partial c}{\partial t} \, dV = \iiint_{V_0} \nabla \cdot (D \nabla c) \, dV + \iiint_{V_0} W_N \, dV ,
\]
which may be presented in the differential form as
\[
\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c) + W_N .
\] (6.40)

Finally, when both hydrodynamic flow and diffusion are important, then the equation of particle drift takes into account both processes
\[
\frac{\partial c}{\partial t} + \nabla \cdot (cu) = \nabla \cdot (D \nabla c) + W_N .
\] (6.41)

6.8 Heat transfer in a flow.

How can we describe heat transfer in a flow \( \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \) by a differential equation taking into account thermal conduction?

Solution:

We consider an arbitrary fixed volume \( V_0 \) shown in Fig. 4.2. The energy balance for the volume is
(heat variation inside the volume) = -(heat flux due to the flow) -
(heat flux due to the thermal conduction) + (sources). \hspace{1cm} (6.42)

Elementary heat inside \( dV \) is determined by temperature \( dH = \rho C_H T \, dV \), where \( C_H \) is specific heat per unit mass (which may be taken constant in many problems) and the combination \( \rho C_H T \) plays the role of heat density. Then total heat inside \( V_0 \) is
\[
H = \iiint_{V_0} \rho C_H T \, dV ,
\]
with
\[
\frac{dH}{dt} = \iiint_{V_0} \frac{\partial}{\partial t} (\rho C_H T) \, dV . \hspace{1cm} (6.44)
\]

Heat flux due to the hydrodynamic flow is similar to the mass flux, Eq. (5.5)
\[
\left( \frac{dH}{dt} \right)_{\text{due to } u} = \iiint_{S_0} \rho C_H T u \cdot dS = \iiint_{V_0} \nabla \cdot (\rho C_H T u) \, dV . \hspace{1cm} (6.45)
\]

Heat flux due to thermal conduction is given by Eq. (5.17)
\[
\left( \frac{dH}{dt} \right)_{\text{due to } \kappa} = -\iiint_{V_0} \kappa \nabla T u \cdot dS = -\iiint_{V_0} \nabla \cdot (\kappa \nabla T) \, dV . \hspace{1cm} (6.46)
\]

In order to describe heat sources we introduce the source density \( W_H \)
\[
\left( \frac{dH}{dt} \right)_{\text{due to sources}} = \iiint_{V_0} W_H \, dV . \hspace{1cm} (6.47)
\]

Then the equation of heat transfer may be presented as
\[
\iiint_{V_0} \frac{\partial}{\partial t} (\rho C_H T) \, dV = -\iiint_{V_0} \nabla \cdot (\rho C_H T u) \, dV + \iiint_{V_0} \nabla \cdot (\kappa \nabla T) \, dV + \iiint_{V_0} W_H \, dV , \hspace{1cm} (6.48)
\]
or
\[
\frac{\partial}{\partial t} (\rho C_H T) + \nabla \cdot (\rho C_H T u) = \nabla \cdot (\kappa \nabla T) + W_H . \hspace{1cm} (6.49)
\]

In the simple case of no flow \( u = 0 \), constant density \( \rho = \text{const} \), constant coefficient of thermal conduction \( \kappa = \text{const} \) and no sources, the above equation reduces to the well-known equation of thermal conduction
\[
\frac{\partial T}{\partial t} = \frac{\kappa}{\rho C_H} \nabla^2 T . \hspace{1cm} (6.50)
\]
6.9 Density variations for a fluid parcel.

The approach of fields deals with parameters defined in a fixed reference frame, e.g. \( \rho = \rho(\mathbf{r}, t), \mathbf{u} = \mathbf{u}(\mathbf{r}, t) \). In that case the derivatives \( \frac{\partial \rho}{\partial t}, \frac{\partial \mathbf{u}}{\partial t} \) define time variation of density (velocity) in a fixed point. How can we find density variations \( \frac{d\rho}{dt} \) for a fluid parcel moving together with the flow?

**Solution:**

Suppose that the parcel trajectory is \( \mathbf{r} = \mathbf{r}(t) \). Then the dependence \( \rho = \rho(\mathbf{r}, t) \) is reduced to a function of one variable \( \rho = \rho(\mathbf{r}(t), t) \), and we have

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \nabla \rho.
\]

By definition

\[
\frac{d\mathbf{r}}{dt} = \mathbf{u},
\]

which leads to

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho.
\] (6.51)

The equation (6.51) is known as substantive (hydrodynamic) derivative. Using the substantive derivative we can calculate the acceleration of a fluid parcel. Applying (6.51) to all components of the fluid velocity we find

\[
\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u},
\]

and similar expressions for \( u_y, u_z \). Multiplying the obtained expressions by \( \hat{e}_x, \hat{e}_y, \hat{e}_z \), respectively, and taking the sum we find the acceleration

\[
\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}.
\] (6.52)

7.1 Faraday's law.

Reduce Faraday's law, Eq. (5.26), to a differential equation.

Solution:

The right-hand side of Faraday’s law
\[
\oint_{c} \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \iint_{S_{0}} \mathbf{B} \cdot d\mathbf{S}
\]
may be transformed by use of the Stokes theorem as
\[
\oint_{c} \mathbf{E} \cdot d\mathbf{r} = \iint_{S_{0}} (\nabla \times \mathbf{E}) \cdot d\mathbf{S},
\]
which leads to
\[
\iint_{S_{0}} (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_{S_{0}} \mathbf{B} \cdot d\mathbf{S}.
\]

Since the surface \( S_{0} \) is arbitrary, then we can present Faraday’s law in the form
\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.
\] (7.1)

In the stationary case (in electrostatics) we have
\[
\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{and, hence}, \quad \nabla \times \mathbf{E} = 0,
\]
which implies that the electric field in electrostatics is potential.

7.2 Magnetic field of currents in electrostatics.

It was obtained experimentally, that circulation of \( \mathbf{B} \) along a closed loop \( C \) is proportional to the total current through any surface \( S \), surrounded by the loop.

\[
\oint_{c} \mathbf{B} \cdot d\mathbf{r} = \mu_{0} I = \mu_{0} \iint_{S} \mathbf{j} \cdot d\mathbf{S}.
\] (7.2)

Reduce Eq. (7.2) to a differential equation.

Solution:

Using the Stokes theorem we transform
\[
\oint_{c} \mathbf{B} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{B}) \cdot d\mathbf{S},
\]
and obtain
\begin{align*}
\oint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \oint_S \mathbf{j} \cdot d\mathbf{S}.
\end{align*}

Since the surface $S$ is arbitrary, then we find

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \quad (7.3)$$

The law (7.3) is not general, but it holds only in a stationary case. In order to write the general relation, Maxwell assumed symmetry of electric and magnetic fields, and completed (7.3) by the so-called displacement current

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.4)$$

The last term in (7.4) is similar to Faraday’s law (7.1), that is, time variation of electric field generates circulation of magnetic field. Equations (6.9), (6.10), (7.1), (7.4) are known as the Maxwell equations:

\begin{align*}
\nabla \cdot \mathbf{E} &= \rho / \varepsilon_0, \quad (7.5\ a) \\
\nabla \cdot \mathbf{B} &= 0, \quad (7.5\ b) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad (7.5\ c) \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.5\ d)
\end{align*}

7.3 Charge conservation and the Maxwell equations.

Using the Maxwell equations derive the equation of charge conservation, (6.30).

Solution:

We take divergence of Eq. (7.5 d) and obtain

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}.$$ 

Substituting Eq. (7.5 a) we reduce it to

$$\mu_0 \nabla \cdot \mathbf{j} + \mu_0 \frac{\partial \rho_\varepsilon}{\partial t} = 0, \quad \text{or} \quad \nabla \cdot \mathbf{j} + \frac{\partial \rho_\varepsilon}{\partial t} = 0$$

which is the equation of charge conservation.
7.4 **Rotor in fluid mechanics. Vorticity**

The vorticity of a fluid velocity is defined as \( \omega = \nabla \times \mathbf{u} \), and \( \Gamma = \oint \mathbf{u} \cdot d\mathbf{r} \) determines velocity circulation along a specified closed loop. Using the Stokes theorem we can see that vorticity is coupled to the elementary circulation as

\[
\Gamma = \oint \mathbf{u} \cdot d\mathbf{r} = \iint (\nabla \times \mathbf{u}) \cdot dS = \iint \omega \cdot dS,
\]

that is \( d\Gamma = \omega \cdot dS \). Physical meaning of vorticity may be understood if we calculate \( \omega \) for a fluid rotating as a rigid body with velocity \( \mathbf{u} = \Omega \mathbf{r} \), that is \( \mathbf{u} = \left( -\Omega y, \Omega x, 0 \right) \) for \( \Omega \) directed along the z-axis, see Example 1.2. Calculating \( \omega \) we find

\[
\omega = \nabla \times \mathbf{u} = \begin{vmatrix}
\hat{e}_x & \hat{e}_y & \hat{e}_z \\
\partial / \partial x & \partial / \partial y & \partial / \partial z \\
-\Omega y & \Omega x & 0
\end{vmatrix} = 2\Omega \hat{e}_z = 2\Omega.
\]

Thus, \( \omega \) is doubled rotational frequency of a fluid parcel.

A flow with zero vorticity, \( \omega = \nabla \times \mathbf{u} = 0 \), is called irrotational, and we may define the velocity potential

\[
\mathbf{u} = \nabla \phi
\]

for such a flow. An irrotational incompressible flow is described by the Laplace equation

\[
\nabla \cdot \mathbf{u} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0. \tag{7.6}
\]

7.5 **Vector potential in fluid mechanics.**

Since an incompressible flow is characterized by zero velocity divergence, \( \nabla \cdot \mathbf{u} = 0 \), then we can introduce a vector potential \( \mathbf{u} = \nabla \times \mathbf{A} \). The vector-potential is not so popular in studies of three-dimensional flows, since in that case we replace an unknown vector-field \( \mathbf{u} \) by another unknown vector-field \( \mathbf{A} \). However, the method of vector-potential is quite useful for two-dimensional problems, for which we can chose \( \mathbf{A} = (0,0,\psi) \). By definition of \( \mathbf{A} \) we have

\[
\mathbf{u} = \nabla \times \mathbf{A} = \hat{e}_x \frac{\partial \psi}{\partial y} - \hat{e}_y \frac{\partial \psi}{\partial x} \tag{7.7}
\]
The scalar field $\psi$ is called a stream function, and it has some interesting properties. Let
us calculate $d\psi$ along a streamline. The equation for a streamline Eq. (1.2) in a two
dimensional flow is

$$\frac{dx}{u_x} = \frac{dy}{u_y}, \quad \text{or} \quad u_x \, dy - u_y \, dx = 0,$$

which may be rewritten with the help of the stream function as

$$\frac{\partial \psi}{\partial y} \, dy + \frac{\partial \psi}{\partial x} \, dx = d\psi = 0.$$  \hspace{1cm} (7.8)

Thus, a stream function is constant along a streamline, and streamlines are also iso-lines
of $\psi = \psi(x, y, t)$.

Let us calculate the flux integral of a two-dimensional vector field of velocity $\mathbf{u}$
through a curve $L$ connecting two streamlines with respective values of the stream
function $\psi_1$ and $\psi_2$

$$\int_L \mathbf{u} \cdot \mathbf{n} \, dr.$$

The normal unit vector in the flux integral may be calculated as

$$\mathbf{n} \, dr = d\mathbf{r} \times \hat{\mathbf{e}}_z = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ dx & dy & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{\mathbf{e}}_x \, dy - \hat{\mathbf{e}}_y \, dx.$$

Then

$$\int_L \mathbf{u} \cdot \mathbf{n} \, dr = \int_L u_x \, dy - u_y \, dx = \int_L \frac{\partial \psi}{\partial y} \, dy + \frac{\partial \psi}{\partial x} \, dx = \int_L d\psi = \psi_2 - \psi_1.$$

Thus the difference $\psi_2 - \psi_1$ determines the flux integral. The physical meaning of the
flux integral is the fluid discharge calculated for a two-dimensional flow.

8.1 Electromagnetic waves in vacuum.

Derive the differential equation describing e-m waves in vacuum.

Solution:

We have neither charges nor currents in vacuum, and the system of Maxwell equations reduces to

\[ \nabla \cdot \mathbf{E} = 0, \]
\[ \nabla \cdot \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \]

Applying rotor to Eqs. (8.3) and taking (8.4) into account we find

\[ \nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \]

On the other hand, according to the standard formulas of nabla-calculations

\[ \nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}. \]

Taking Eq. (8.1) into account we obtain the wave equation

\[ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \]

where \( c = 1/\sqrt{\varepsilon_0 \mu_0} \) is the speed of light in vacuum. Using Fourier representation we may describe a wave with the help of functions like \( \cos(kx \pm \omega t) \), \( \sin(kx \pm \omega t) \), \( \exp(ikx \pm i\omega t) \), where \( k = 2\pi/\lambda \) is the wave number, \( \lambda \) is the wavelength, and the frequency \( \omega \) depends on the wave number according to the dispersion relation \( \omega = \omega(k) \).

The dispersion relations may be different for different types of waves. Such a relation contains most of the information about wave properties, and, typically, the main purpose of the wave studies is to find the dispersion relation. In the present example we find the dispersion relation for electromagnetic waves in vacuum.

In the three-dimensional geometry the Fourier representation involves functions like \( \exp(ik \cdot r \pm i\omega t) \), where \( k \) is a wave vector. The sign \( \pm \) under the exponent
corresponds to the waves propagating to the right and to the left, respectively. Below we will keep the minus sign. Let us introduce the “running variable” \( \xi = \mathbf{k} \cdot \mathbf{r} - i\omega t \), so that a Fourier harmonic for a wave may be written as

\[
\mathbf{E} = a \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t),
\]

where \( a \) is the amplitude. Substituting Eq. (8.7) into the wave equation (8.6) we find

\[
\frac{\partial}{\partial t} \mathbf{E} = \frac{\partial}{\partial t} a \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) = \frac{\partial}{\partial t} a \exp(i\xi) = \frac{\partial}{\partial t} \left( -i\omega a \exp(i\xi) \right),
\]

\[
\frac{\partial^2}{\partial t^2} \mathbf{E} = (-i\omega)^2 a \exp(i\xi) = -\omega^2 a \exp(i\xi),
\]

\[
\nabla^2 \mathbf{E} = a \nabla^2 \exp(i\xi) = a \nabla \cdot (\nabla \exp(i\xi)) = a \nabla \cdot (\exp(i\nabla \cdot \xi) = a \nabla \cdot (i\mathbf{k} \exp(i\xi)) = a \nabla \cdot (i\mathbf{k} \exp(i\xi)) = (i\mathbf{k})^2 a \exp(i\xi) = -k^2 a \exp(i\xi),
\]

and

\[
-\frac{\omega^2}{c^2} a \exp(i\xi) + k^2 a \exp(i\xi) = 0,
\]

which leads to the dispersion relation

\[
\omega^2 = c^2 k^2.
\]

Other interesting formulas of \( \nabla \)-calculations for the Fourier harmonics are

a) \( \nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} \),

b) \( \nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} \).

Indeed,

\[
\nabla \cdot \mathbf{E} = \nabla \cdot (a \exp(i\xi)) = a \cdot \nabla \exp(i\xi) = a \cdot i\mathbf{k} \exp(i\xi) = i\mathbf{k} \cdot \mathbf{E},
\]

and

\[
\nabla \times \mathbf{E} = \nabla \times (a \exp(i\xi)) = (\nabla \exp(i\xi)) \times a = i\mathbf{k} \times a \exp(i\xi) = i\mathbf{k} \times \mathbf{E}
\]

Example: Show that electromagnetic waves in vacuum are “perpendicular” with \( \mathbf{k} \perp \mathbf{E} \), \( \mathbf{k} \perp \mathbf{B} \), \( \mathbf{E} \perp \mathbf{B} \).

Solution:

According to Eq. (8.1) we have in vacuum \( \nabla \cdot \mathbf{E} = 0 \). Then \( \nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} = 0 \) and \( \mathbf{k} \perp \mathbf{E} \). In a similar way we can show that \( \nabla \cdot \mathbf{B} = i\mathbf{k} \cdot \mathbf{B} = 0 \) and \( \mathbf{k} \perp \mathbf{B} \). Finally, we have from Eq. (8.3)
\[ \mathbf{i} \mathbf{k} \times \mathbf{E} = i \omega \mathbf{B}, \]

that is
\[ \mathbf{B} = \frac{k}{\omega} \mathbf{E}. \]

9. Other integral theorems.

9.1 Hydrostatics.

Derive the differential equation describing pressure distribution for a fluid at rest in a gravitational field.

Solution:

We consider an arbitrary volume in the fluid similar to that shown in Fig. 4.2. The fluid is at rest, which means that the fluid velocity is zero \( \mathbf{u} = 0 \) and the net force acting on the fluid inside the volume is zero too. The net force results from the gravity force and the pressure force, which leads to the equation of force balance
\[ \mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{pressure}} = 0. \]

The gravity force is calculated according to Eq. (2.5), and the pressure force is determined by Eq. (5.24), which leads to
\[ \iiint_{V_0} \mathbf{g} \rho \, dV - \iiint_{S_0} P \, dS = 0. \]

Using the respective integral theorem we reduce the surface integral to the volume integral
\[ \iiint_{S_0} P \, dS = \iiint_{V_0} \nabla P \, dV \]
and obtain the integral equation
\[ \iiint_{V_0} (\mathbf{g} \rho - \nabla P) \, dV = 0. \]

Since the equation holds for an arbitrary volume, then we can write it in a differential form
\[ \mathbf{g} \rho = \nabla P. \]

(9.1)
In the everyday situation we have approximately constant gravity acceleration \( \mathbf{g} = (0; 0; -g) \), where we have chosen \( z \)-axis pointing upwards. In that case the equation of hydrostatics takes the form

\[
\frac{\partial P}{\partial z} = -\rho g, \quad \frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0. \tag{9.2}
\]

In the case of incompressible fluid with \( \rho = \text{const} \) the solution to Eq. (9.2) is

\[
P = -\rho g z + P_0. \tag{9.3}
\]

An interesting consequence of the solution Eq. (9.3) is that pressure at the bottom of a vessel does not depend on the vessel form, but only on the fluid depth. As a result, pressure at the bottom of the tanks shown in Fig. 9.1 is the same.

![Fig. 9.1. Tanks of water with equal pressure at the bottom](image)

9.2 The Archimedes force.

Find the pressure force acting on a body immersed in an incompressible fluid (the Archimedes force).

Solution:

The pressure force is determined by Eq. (5.24), which may be transformed into the volume integral

\[
\mathbf{F} = -\iiint_{S_0} P \, d\mathbf{S} = -\iiint_{V_0} \nabla P \, dV.
\]
In that case $\nabla P$ stands for virtual pressure distribution, which could take place if the fluid filled the whole space. The virtual pressure distribution obeys the hydrostatic equation Eq. (9.1) with $\rho$ standing for the fluid density. On the other hand, it is obvious that a body at rest does not change the pressure distribution. Then the pressure force acting on the body is

$$F = -\iiint_V \nabla P \, dV = -\iiint_V \rho \, dV = -g \rho \iiint_V dV = -\rho g V_{\text{body}}.$$  

Thus, the fluid pressure force on a body is equal to the weight of the fluid, which could fill the volume of the body. The force is directed upwards.

### 9.3 Force on a magnetic dipole.

We consider a magnetic dipole, which is, actually, a small loop $L$ of wire with current $I$, with surface area $dS = \hat{n} dS$, see Fig. 9.3. The magnetic moment of the dipole is defined as

$$\mathbf{\mu} = I dS,$$

The magnetic dipole is placed in a magnetic field $\mathbf{B}(\mathbf{r})$.

![Fig. 9.3. Magnetic dipole placed in a magnetic field](image)

**Example:** Find the force acting on the dipole.

**Solution:**

We use Ampere’s law Eq. (4.1) for the force acting on an element of the wire

$$d\mathbf{F} = I \, d\mathbf{r} \times \mathbf{B}.$$  

The total force is

$$\mathbf{F} = I \oint_L d\mathbf{r} \times \mathbf{B}.$$  

Using the integral theorems we can transform the above formula to
\[
F = I \int_L \vec{d}r \times \vec{B} = I \iint_{dS} (dS \times \nabla) \times \vec{B} = I \iint_{dS} (\hat{n} \times \nabla) \times \vec{B} dS.
\]

Standard \ \nabla \cdot \text{-calculations lead to}
\[
(\hat{n} \times \nabla) \times \vec{B} = \nabla (\hat{n} \cdot \vec{B}) - \hat{n} (\nabla \cdot \vec{B}) = \nabla (\hat{n} \cdot \vec{B}),
\]
since \ \nabla \cdot \vec{B} = 0. Then
\[
F = I \iint_{dS} \nabla (\hat{n} \cdot \vec{B}) dS \approx \nabla \left( \int_{dS} \vec{B} \cdot \hat{n} \right) = \nabla (\mu \cdot \vec{B}).
\]