# GAINING MATHEMATICAL UNDERSTANDING:

## The effects of creative mathematical reasoning and cognitive proficiency

Bert Jonsson, Johan Lithner, Carina Granberg, Mathias Norqvist, Yvonne Liljeqvist, Tony Qwillbard, Linus Holm

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# ACTIVE LEARNING: TWO MAIN APPROCHES

1. Retrieval practice (testbaserat lärande)
2. Creative mathematical reasoning

# BASED ON A THEORETICAL FRAMEWORK

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational studies in mathematics*, *67*(3), 255-276.

Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM*, *49*(6), 937-949.

# INVESTIGATED THROUGH SEVERAL EMPIRICAL STUDIES


Jonsson, B Granberg, G, Lithner, J.(2020). Gaining mathematical understanding: The effects of creative mathematical reasoning and cognitive proficiency. *Frontiers in psychology (pending revision)*

Norqvist, M., Jonsson, B., Lithner, J., Qwillbard, T., & Holm, L. (2019). Investigating algorithmic and creative reasoning strategies by eye tracking. *The Journal of Mathematical Behavior*. doi:https://doi.org/10.1016/j.jmathb.2019.03.008

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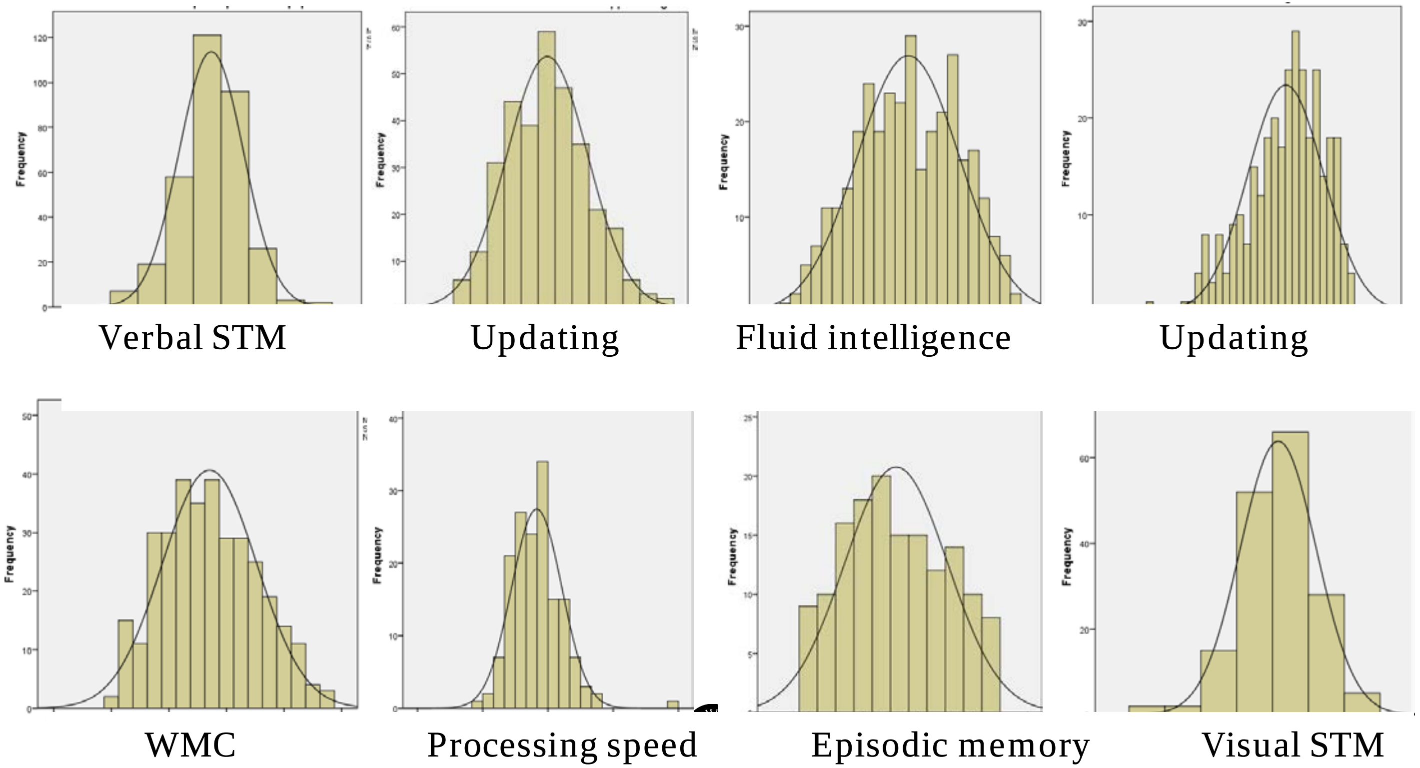
# INDIVIDUAL DIFFERENCES IN COGNITION



# FRAMWORK:

## CREATIVE AND ALGORITHMIC (IMITATIVE) REASONING

•Aim: Develop teaching models that are more efficient for learning than common imitative models.

- The teacher and/or textbook **describes the algorithm and explains it**, which seems to be many teachers' ideal
  Denoted as Algorithmic reasoning (AR)

- The student constructs the solution method: the student construct the new knowledge herself
  Denoted as Creative Mathematical Reasoning (CMR)

- **Algorithmic reasoning (AR)**

*Solutions are provided*

- An algorithm is a finite sequence of instructions which allows one to find a definite result for a given class of problems (y=3x+1)

- It has high reliability and can rapidly produce an answer

- It can be used without (almost) any understanding of the task.

**Creative mathematically reasoning (CMR).**

*Constructing the solutions*

- *Creativity;* a new reasoning (new to the reasoner) sequence is created, or a forgotten one is re-created

- *Plausibility;* there are arguments supporting the strategy choice and/or strategy implementation explaining why the conclusions are true or plausible;

- *Anchoring*; the arguments are anchored in the intrinsic mathematical properties of the components that are involved in the reasoning required to solve the problem
  - e.g 6/9 is a smaller proportion than 3/4

## Algorithmic Reasoning (AR)

When squares are put in a row it looks like the figure to the right. 13 matches are needed for four squares:

If $x$ is the number of squares then the number of matches $y$ can be calculated by the function $y=3x+1$

*Example*: If 4 squares are put in a row then
$y=3x+1=3\cdot4+1=13$ matches are needed.

**How many matches are needed to get 6 squares in a row?**

## Creative Mathematical Reasoning (CMR)

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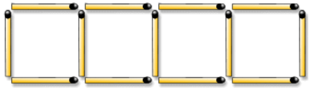
# Practice tasks

**A) Practice AR-task, method provided**



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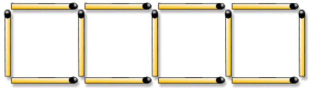
If x is the number of squares then the number of matches y could be calculated by the function

$y = 3x + 1$

*Example*: If 4 squares are put in a row then $y = 3x + 1 = 3 \cdot 4 + 1 = 13$ matches are needed

**How many matches are needed to get 100 squares in a row?**

**B) Practice CMR-task, constructing method**



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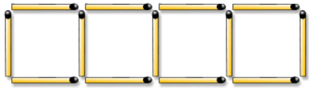
**C) Practice CMR-task, constructing formula**



When squares are put in a row, it
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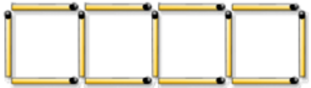
If x is the number of squares in a row and y is the number of matches needed to build the squares.

**How could you describe y as a function of x?**

# Test tasks

**A) Posttest practice task**



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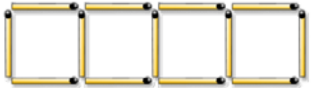
**C) Posttest formula practice task**



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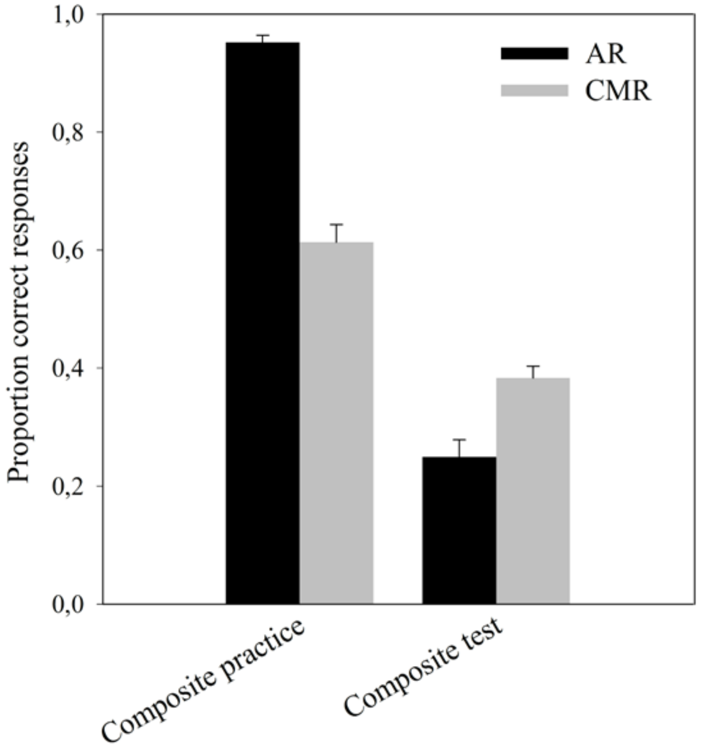


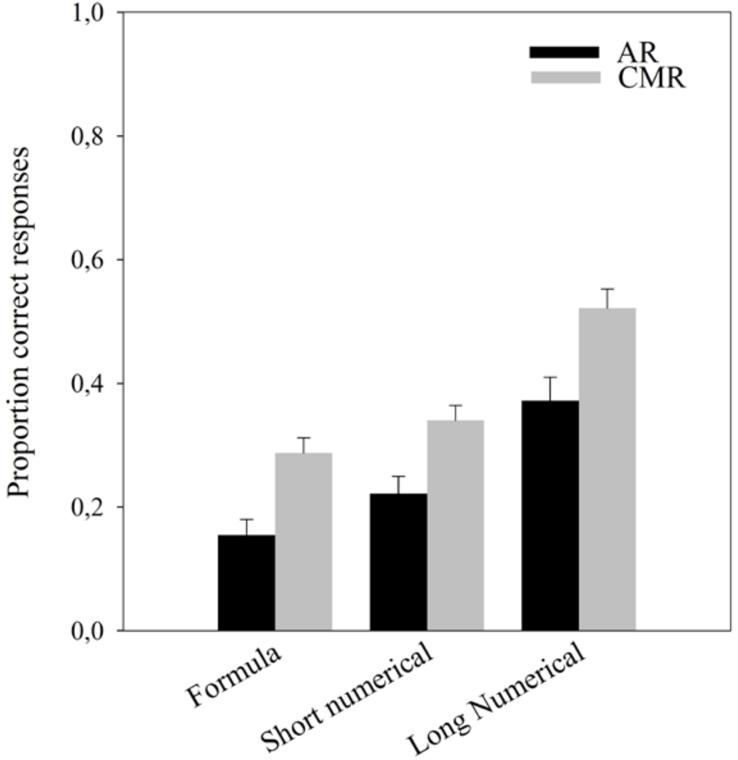


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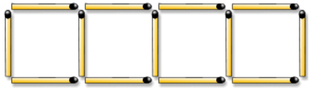

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*Example*: If 4 squares are put in a row then $y = 3x + 1 = 3 \cdot 4 + 1 = 13$ matches are needed

**How many matches are needed to get 100 squares in a row?**

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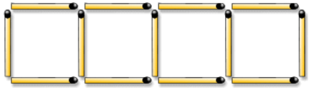

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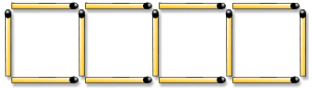

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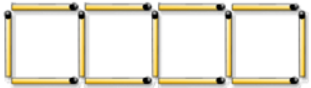

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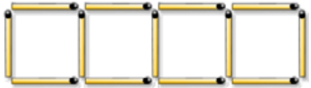
**C) Posttest formula practice task**



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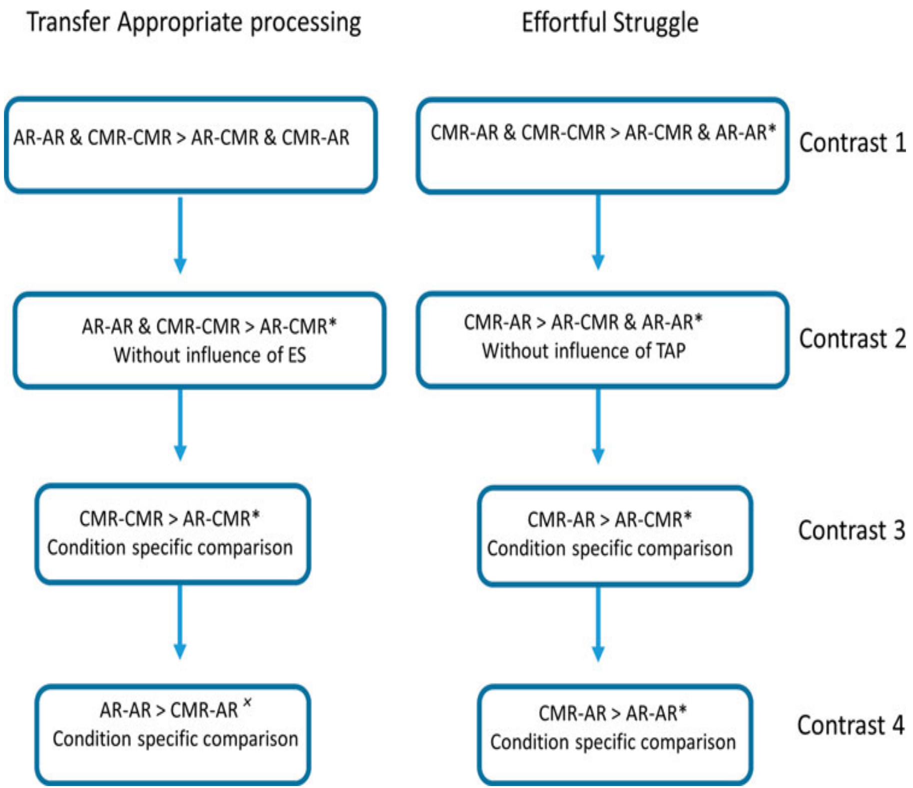


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## ***Transfer-appropriate processing***

The average effect size (Cohen's *d*) was found to be 0.27, which is considered as a small effect size.

## ***Effortful struggle***

The average effect size (Cohen's *d*) was found to be 1.34, which is clearly above 0.8 – the margin for a large Cohen's *d* effect siz

**A) Posttest practice task**

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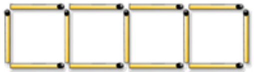
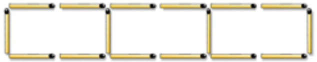


**How many matches are needed to get 100 squares in a row?**

**B) Posttest transfer task**

When rectangles are put in a row, it looks like the figure on the right, 16 matches are needed for three rectangles.



**How many matches are needed to get 100 rectangles in a row?**

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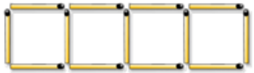

If x is the number of squares in a row and y is the number of matches needed to build the squares.

**How could you describe y as a function of x?**

**D) Posttest formula transfer task**

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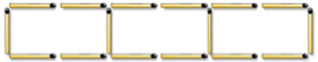

If x is the number of rectangles in a row and y is the number of matches needed to build the rectangles.

**How could you describe y as a function of x?**

Can practicing with CMR tasks elicit reasoning that can be transferred to different tasks that require similar general solution ideas but different specific solution methods?

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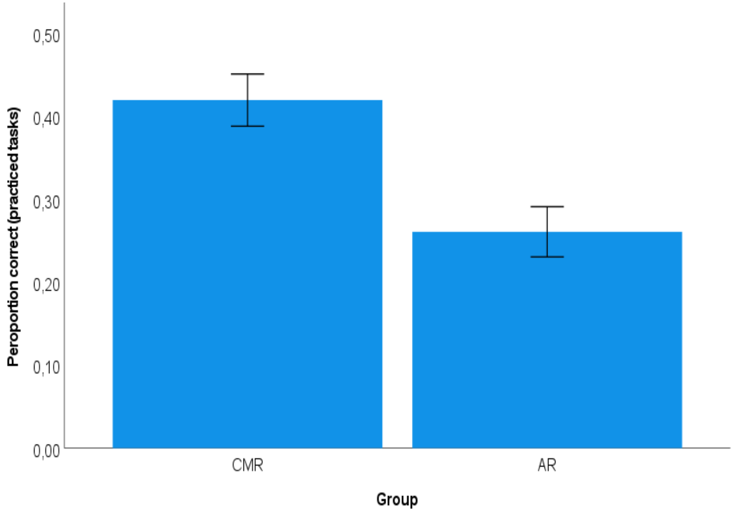
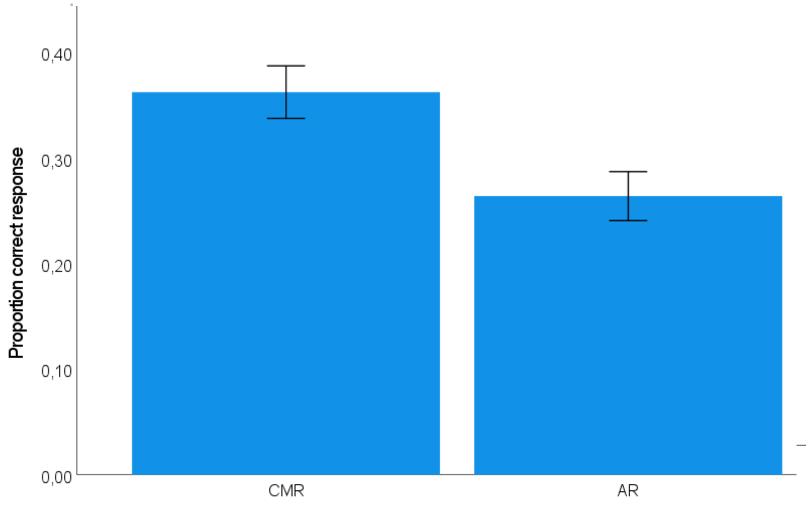
# DIFF CMR -AR

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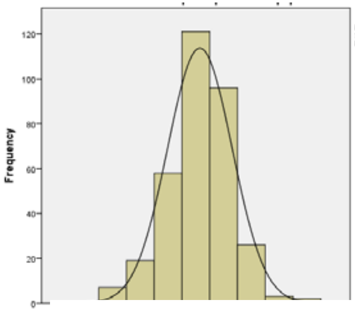
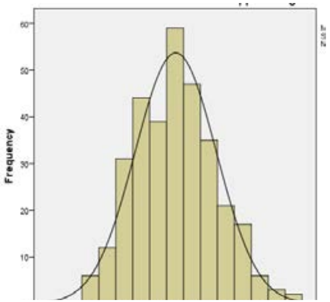
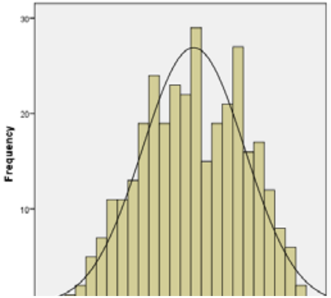
# INDIVIDUAL DIFFERENCES IN COGNITION



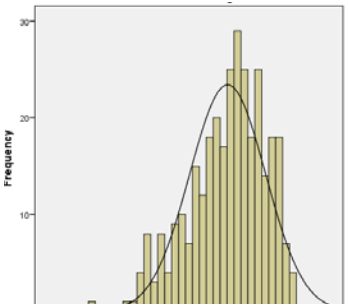
Verbal STM



Updating



Fluid intelligence



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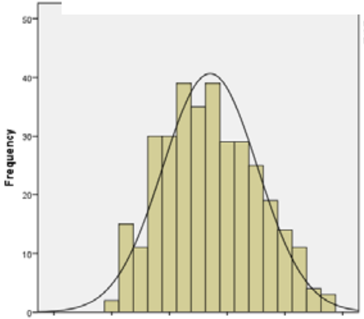
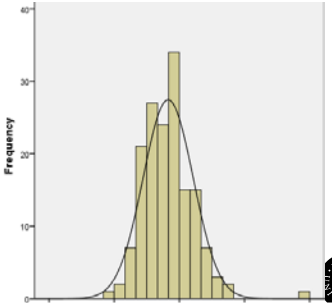
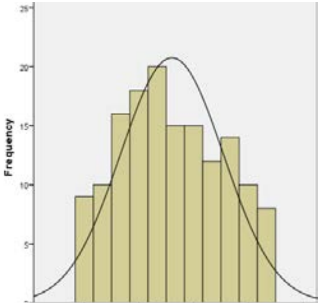


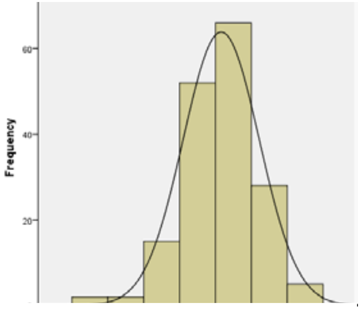
WMC



Processing speed



Episodic memory



Visual STM

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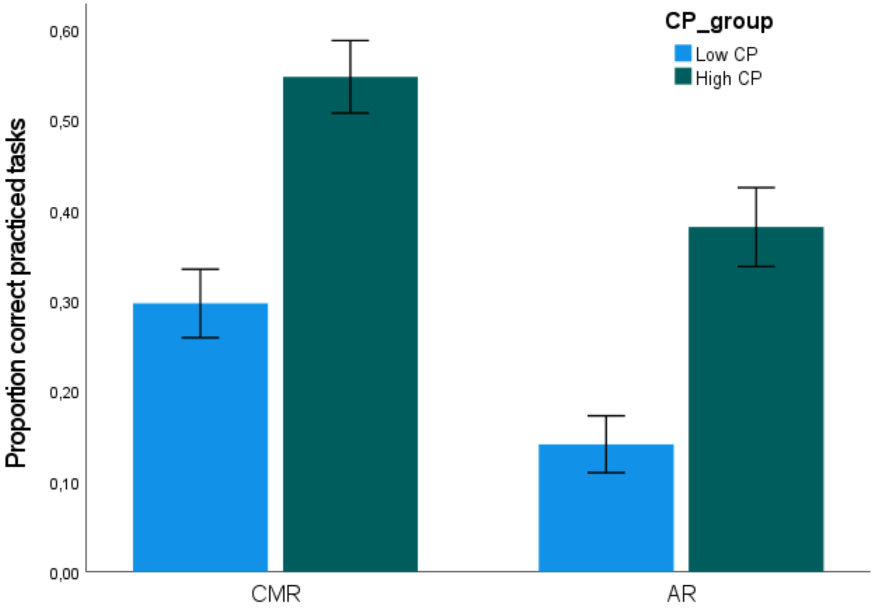


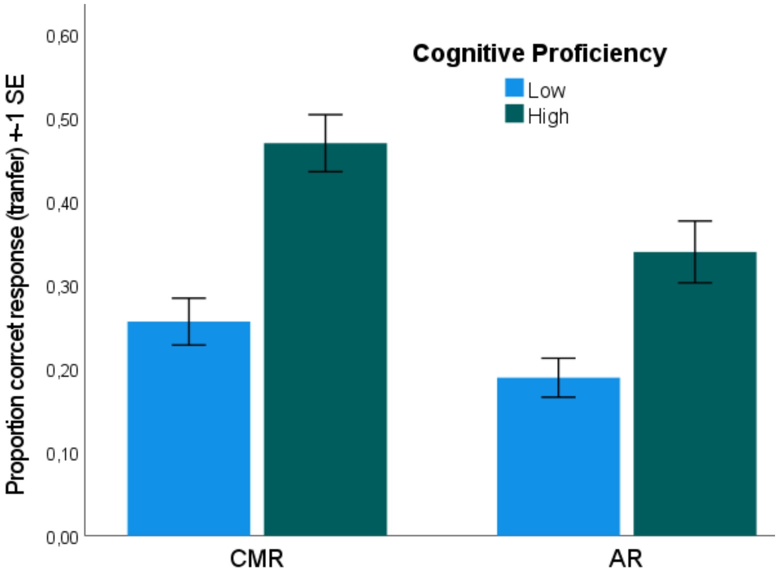


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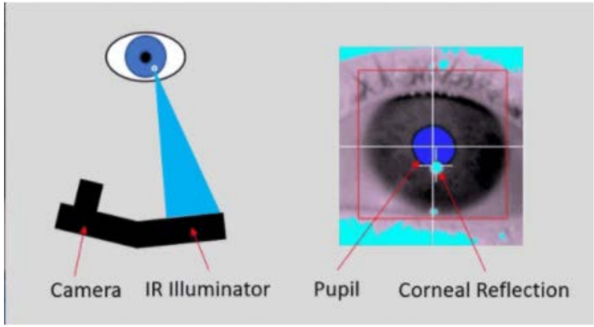
## Method: Corneal eye reflection technique

- A camera focuses on one or both eyes

- Uses infrared light to create corneal reflections (CR).
- Fixations are defined as a period between saccades, excluding blink of the eye

- Only fixations with durations longer than 50 ms were considered for analysis



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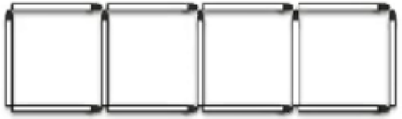
00000626 ms

Kvadrater i en rad sätts samman som i figuren till höger. Till 4 kvadrater i rad behövs 13 tändstickor.



Om $x$ är antalet kvadrater som ska läggas i rad så kan man beräkna antalet tändstickor $y$ med funktionen till höger.

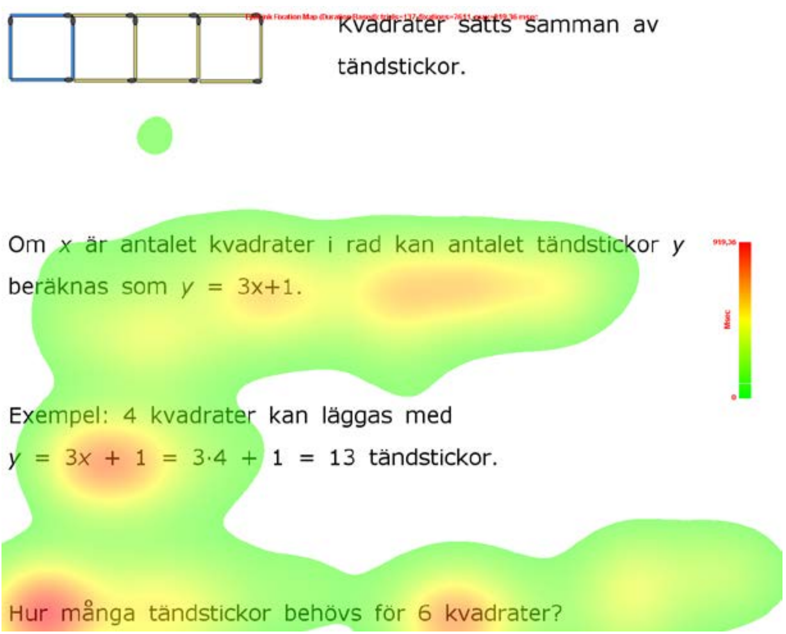
$$\boxed{y = 3x + 1}$$

*Exempel*: Om 4 kvadrater ska läggas i rad behövs $y=3x+1=3\cdot4+1=13$ tändstickor.

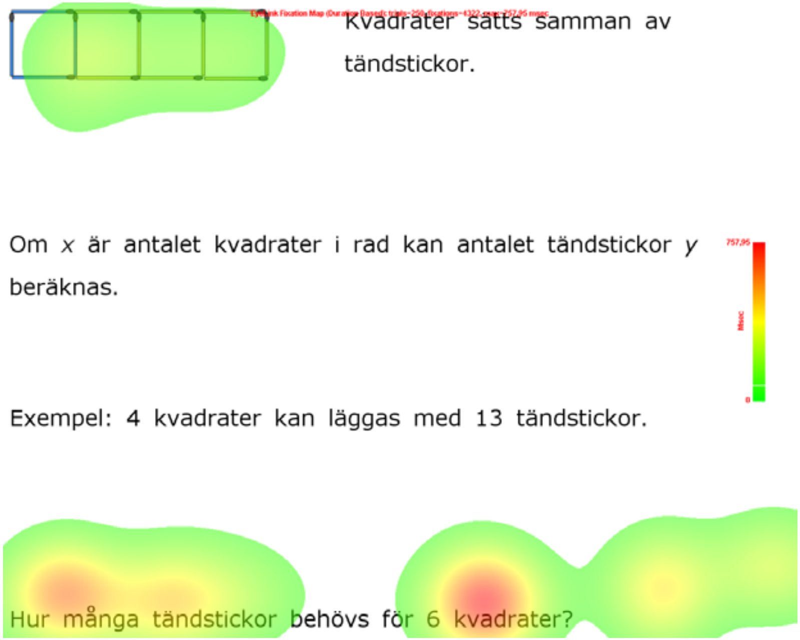
**Hur många tändstickor behövs för att få 100 kvadrater i rad?**

# Heatmaps

## AR




## CMR

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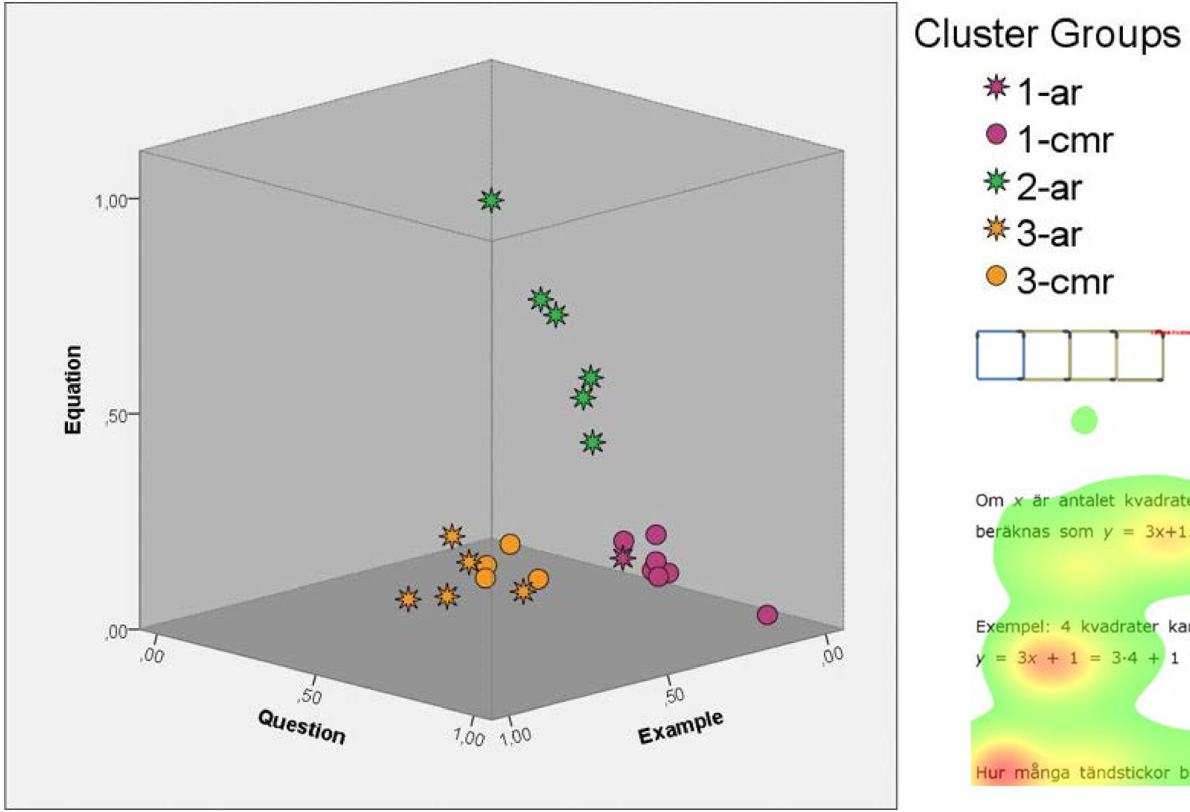
## Cluster analysis

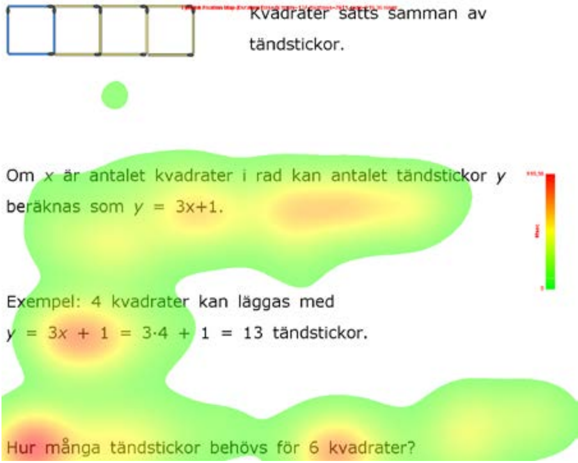
- Group objects/individuals/performance so that those that are more similar ends up in the same grouping- so called clusters

- **Hierarchical cluster analysis**: all the information (eye fixations) from all the areas of interests are extracts and clusters of eye fixations are formed.

- Variance within clusters are minimized and variance between cluster are maximized and through at iterative process amalgamated into clusters of increasing dissimilarities(Ward, 1963).

# Average eye fixations



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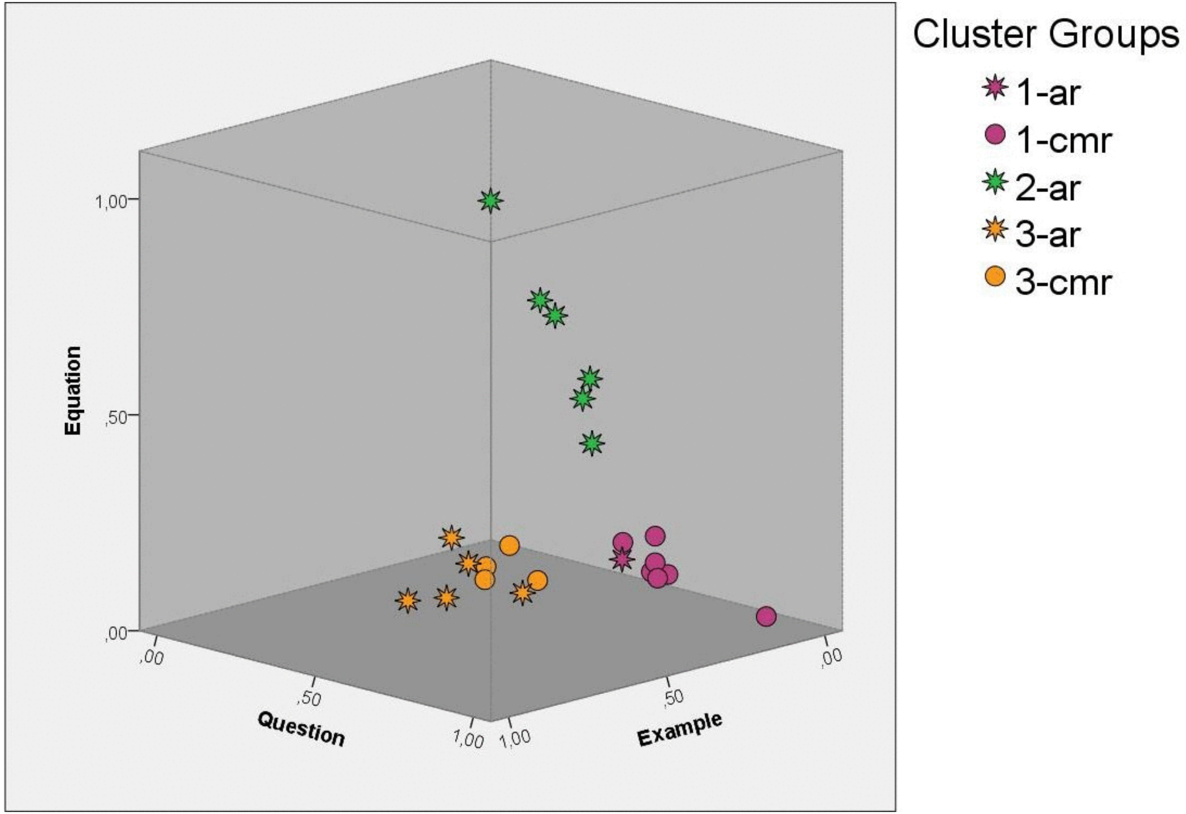


# IN SUM

- CMR an active approach more beneficial than the more passive AR approach
- Cognitive ability does play a role, but is independent of learning condition CMR/AR
- Eye tracking analyses indicate that cognitively weaker tend to focus on the irrelevant information
- Effortful struggle seems to be important- likely associated with the construction of tasks

- **Consequences**
  - Let them struggle by constructing
  - "The more the merrier" in terms of information is not always the best
    - Especially for cognitively less proficient students

# IN SUM

- What we don’t know
  - The effects on Special education needs students
  - Long-term effects
  - What characterize those students that benefit relative those that don’t benefit
    - Effects of personality

# THANK YOU FOR YOUR ATTENTION

bert.jonsson@umu.se

https://www.researchgate.net

https://scholar.google.com/

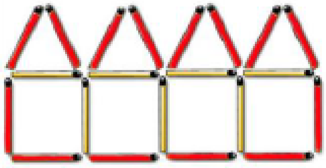
### D) AR-practice task

With 6 matches, you can build a house. The houses can be put together into a row of houses. The outer edge is built of red matches

If x is the number of houses, the number of red matches y can be calculated by the formula $y = 3x + 2$

*Example*: If 4 houses are put together, $y = 3 \cdot 4 + 2 = 14$ red matches are needed

**How many red matches are needed to put 100 houses together?**



### E) CMR-practice task

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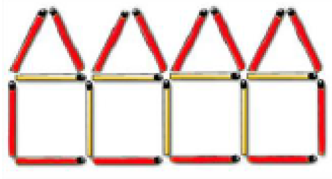
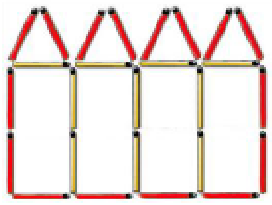
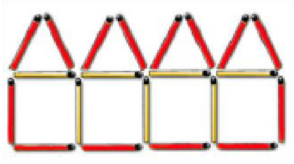


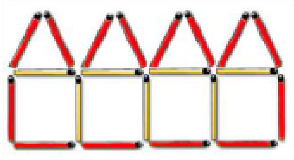
### F) Transfer Posttest task

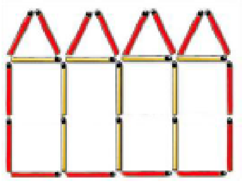
With 8 matches, you can build a two-stored house. The houses can be put together into a row of houses. The outer edge is built of red matches. If 4 two-stored houses are put together in a row, 14 red matches are needed

**How many red matches are needed to put 100 two-stored houses together?**



| D) AR-practice task | |
|---|---|
| With 6 matches, you can build a house. The houses can be put together into a row of houses. The outer edge is built of red matches<br><br>If x is the number of houses, the number of red matches y can be calculated by the formula $y = 3x + 2$<br><br>*Example*: If 4 houses are put together, $y = 3 \cdot 4 + 2 = 14$ red matches are needed<br><br>**How many red matches are needed to put 100 houses together?** |  |

| E) CMR-practice task | |
|---|---|
| With 6 matches, you can build a house. The houses can be put together into a row of houses. The outer edge is built of red matches. If 4 houses are put together in a row, 14 red matches are needed<br><br>**How many red matches are needed to put 100 houses together?** |  |

| F) Transfer Posttest task | |
|---|---|
| With 8 matches, you can build a two-stored house. The houses can be put together into a row of houses. The outer edge is built of red matches. If 4 two-stored houses are put together in a row, 14 red matches are needed<br><br>**How many red matches are needed to put 100 two-stored houses together?** |  |