Problems to start with

1. Equation $x^3 - 6x^2 + 3x + 4 = 0$ has three distinct roots, which we denote by a, b and c. Calculate the value of $a^2 + b^2 + c^2$. (Hint: **do not** attempt to find an explicit expressions for a, b and c - you are very likely not going to succeed).

2. Equation $x^4 + x^3 - 10x^2 + 1$ has four distinct roots, which we denote by a, b, c and d. Calculate the value of

   $$a^2b + ab^2 + a^2c + ac^2 + a^2d + ad^2 + b^2c + bc^2 + b^2d + bd^2 + c^2d + cd^2.$$

   (Comment: Suppose the roots are instead denoted by x_1, x_2, x_3 and x_4. Then the above expression is the so-called 'symmetric sum' of $x_i^2 x_j$ - that is, it contains all the $12 = 4 \cdot 3$ terms of the form $x_i^2 x_j$ for any choices of $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.)

3. Suppose that the constants a, b and c are chosen so that the polynomial $p(x) = x^3 + ax^2 + bx + c$ satisfies $p(0) = 0$, $p(1) = 1$ and $p(2) = 2$. Find the value of $p(3)$.

4. Now suppose that the constants $a_0, a_1, \dots, a_7$ are chosen so that the polynomial $p(x) = x^8 + a_7x^7 + a_7x^7 + \dots + a_1x + a_0$ satisfies $p(i) = i$ for every $i \in \{0, 1, \dots, 7\}$. Find the value of $p(8)$.

   (**Note:** if your solution to the previous problem involved heavy calculations, then you are not going to be able solve this problem by using similar methods).

More difficult problems

5.
   - Let b and c be constants, and let $P(x) = x^3 - 3bcx + b^3 + c^3$. Prove that $P(-b - c) = 0$.
   - Deduce that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$ for all $a, b, c \in \mathbb{R}$
   - Prove that $a^3 + b^3 + c^3 \geq 3abc$ whenever a, b, c are positive real numbers.

6. Suppose that a, b and c are real numbers which satisfy the conditions $a^2 + b^2 = 2c^2$, $a \neq b$, $c \neq -a$ and $c \neq -b$. Prove that

   $$\frac{(a + b + 2c)(2a^2 - b^2 - c^2)}{(a - b)(a + c)(b + c)} = 3.$$

7. Solve system of equations $x^2 + y^2 = 6z$ and $y^2 + z^2 = 6x$ and $z^2 + x^2 = 6y$.

8. Find all polynomials p which satisfy the condition $(x-10)p(x) = xp(x-1)$ for every $x \in \mathbb{R}$.