# Problem set for the September 24, 2025 session

## 1 Example problem

1. A two-player game is played on a $5 \times 5$ grid. A token starts on the bottom left-hand corner of the grid. On each turn, a player can move the token one or two grid-squares to the right, or to the leftmost square of the row immediately above the token's current row in the grid. The last player who is able to move wins. Which player has a winning strategy?

## 2 Problems to attempt during the session

1. There are 2025 coins on the table. Anna and Berit take turns removing either 1, 3 or 6 coins from the stack of coins. The player who removes the last coin from the table wins. Which of the players has a winning strategy? *(Note: if there are, for example, exactly* 4 *coins on the table, one can only remove* 1 *or* 3 *of them).*

2. Anna and Berit take turns eating pieces from a $m \times n$ rectangular chocolate bar with $m, n \geq 2$ according to the following rules. The chocolate bar is placed on a table, and a jelly bean is placed on its bottom-left corner square. A move consists of removing an arbitrary non-empty $a \times b$ rectangle from the top-right corner of the chocolate bar. The player who removes the bottom-left square (the one with the jelly bean) loses. Which of the players has a winning strategy?

   *(More precisely: suppose that the chocolate squares are indexed as* $(i, j)$ *for* $1 \leq i \leq m$ *and* $1 \leq j \leq n$ *at the start. A move consists of choosing a square* $(a, b)$ *that is not removed yet, and removing all the squares* $(x, y)$ *with* $a \leq x \leq i$ *and* $b \leq y \leq j$ *that have not yet been removed. The person who removes the square* $(1, 1)$ *loses. Who has winning strategy?)*

3. Anna and Berit play the following game. On her first turn, Anna gives Berit the number 2. Whenever one of the players gives number $n$ to the other player, the other player must respond with an integer $m$ that satisfies the condition $n < m < 2n$ (e.g. when Anna gives 2, Berit must return 3, as 3 is the only integer strictly between 2 and 4; and on her next move, Anna can thus return either 4 or 5). The player who first returns a number strictly greater than 100 to the other player loses. Which of the players has a winning strategy?

# 3 Problems to try at home

1. The number 2025 is written on a whiteboard. Anna and Berit play the following game: on each turn, a player chooses a divisor $d$ of the number $n$ that is currently written on the board, and replaces the number $n$ by $n - d$. Anna and Berit play on alternating turns, with Anna starting. The player who writes down 1 at the end of her turn wins. Which of the players has a winning strategy?

2. Show that Hex is a player 1 win.

   *Recall that the game of Hex is played on a rhombus-shaped $n \times n$ board with hexagonal tiles. Anna and Berit take turns claiming tiles from the board by placing Alabaster and Black stones respectively. Anna wins if she manages to claim a path of hexagonal tiles that crosses the board from left to right, while Berit wins if she manages to claim a path of hexagonal tiles from top to bottom. The game of Hex has amongst other things featured in the film 'A beautiful mind' about the mathematician John Nash Jr. See below for a picture of a Hex game on an $11 \times 11$ board, where Anna has won.*

3. Let $n$ be a positive integer. All positive divisors of $n$ are written on the board (including 1 and $n$). Anna and Berit then play the following game, with Anna starting. On each turn, a player chooses an integer $d$ that is still written on the board, and erases all the multiples of $d$ from the board (i.e. those integers on the board that are of the form $d$, $2d$, $3d$, ...). The player who erases the number 1 from the board loses the game. Prove that Anna has a winning strategy for all $n > 1$.

4. Prove that noughts and crosses ('tripp trapp trull' in Swedish) is always a draw on an $n \times n$ board when $n \geq 3$. (*Note that on the $n \times n$ board a winning set is a horizontal, vertical or diagonal line of $n$ symbols.*)

**Questions?** Email `victor.falgas-ravry@umu.se` if you are not sure you understand a question, and bring your solutions to the next session if you want them checked.