UMEÅ UNIVERSITY

Major breakthroughs in Ramsey Theory

(Stora genombrott inom Ramseyteorin)

Credit: 10 ECTS (consists of two independent 5 ECTS modules)

Course coordinator:
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Course Period:
February – June 2026

Main field of study and progress level:
Mathematics, PhD

Prerequisites:
Students should possess mathematical maturity and a solid background in combinatorics, probability theory and graph theory. Previous exposure to probabilistic combinatorics, random graphs, and Ramsey theory is highly desirable.

Objective
This course will cover recent major breakthroughs in Ramsey theory, namely exponential improvements on both upper and lower bounds on the Ramsey numbers.

Contents:
The Ramsey number R(s,t) is the least n such that any 2-colouring of the edges of the complete graph on n vertices contains either a complete subgraph on s vertices all of whose edges are in colour 1 or a complete subgraph on t vertices all of whose edges are in colour 2. The existence of Ramsey numbers follows from a 1930 paper by the celebrated economist, philosopher and mathematician Frank Plumpton Ramsey.

Erdős and Szekeres proved in 1935 that R(s,t) is at most the binomial coefficient s+t-2 choose s-1 using an inductive argument based on the pigeon-hole principle. In particular, for the diagonal Ramsey number R(t,t), this gave an upper bound of 4 to the power t+o(t). In the other direction, a seminal 1945 paper of Erdős based on random colourings gave a lower bound of sqrt(2) to the power t+o(t); this highly influential paper was, in fact, the introduction of the celebrated probabilistic method in combinatorics.

These two approaches left an exponential gap, and making any improvement at either end has been a major goal of extremal combinatorics in the decades since these two papers. For all their simplicity, for more than seventy years improvements to the upper and lower bounds on Ramsey numbers were confined to the o(t) term in the exponent — and even these were either the results of major innovations such as the discovery of the local lemma in the 1970s or the exploration of quasi-randomness and its interplay with structure from the late 1980s to the late 2000s.

However, in the past few years, stunning progress has been made. First, Campos, Griffiths, Morris and Saharsrabudhe obtained an exponential improvement in the upper bounds, using a completely new approach. Their ideas were pushed and refined in a series of follow-up papers. Secondly, MA, Shen and Xie used high-dimensional random geometric graphs to obtain exponential improvements on lower bounds on the off-diagonal Ramsey numbers.

In this course, the student will go through these landmark results and the necessary theory that one must build up to understand their proofs and ideas.

Form of instruction:
The teaching methods are self-study combined with scheduled meetings to discuss course content. The primary reading materials for the course are the papers listed under the literature section, together with Alon and Spencer's The Probabilistic Method and Mathew Penrose's Random geometric graphs as references on the underlying theoretical tools.

Examination:
The examination consists of a series of oral presentations on topics selected by the course coordinator.

Literature:

1. Noga Alon and Joel Spencer. *The probabilistic method*. John Wiley & Sons, 2016.
2. Paul Balister, Béla Bollobás, Marcelo Campos, Simon Griffiths, Eoin Hurley, Robert Morris, Julian Sahasrabudhe, and Marius Tiba. *Upper bounds for multicolour Ramsey numbers*. Preprint (2024), available on arXiv at https://arxiv.org/abs/2410.17197.
3. Marcelo Campos, Simon Griffiths, Robert Morris, and Julian Sahasrabudhe. *An exponential improvement for diagonal Ramsey*. Preprint (2023), available on arXiv at https://arxiv.org/abs/2303.09521.
4. Jie Ma, Wujie Shen, and Shengjie Xie. *An exponential improvement for Ramsey lower bounds*. Preprint (2025), available on arXiv at https://arxiv.org/abs/2507.12926.
5. Mathew Penrose. *Random geometric graphs*. Vol. 5. OUP Oxford, 2003.
6. Yuval Wigderson. *Upper bounds on diagonal Ramsey numbers*. Bourbaki seminar 1230 (2024), available on arXiv at https://arxiv.org/abs/2411.09321.