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Introduction to Symmetries of Differential Equations

Credit: 4 ECTS

Course coordinator:
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Course Period:
February – April 2023

Main field of study and progress level:
Mathematics, PhD

Prerequisites:
Students should have a basic knowledge of linear algebra, multivariable calculus, and differential geometry. In particular, students are expected to be familiar with the theory of functions and vector fields on manifolds, as well as the foundations of the theory of Lie groups.

Objective
This course aims to introduce the geometrical foundations of Lie point transformations and symmetries of ordinary and partial differential equations (ODEs and PDEs, respectively), as well as a solid foundation for analysing and classifying differential equations with respect to their symmetries.

Contents:
Symmetries of a differential equation are transformations that map solutions to other solutions. Of particular importance is the case of continuous groups of symmetries, which provided the original motivation for the development of Lie group theory. The analysis of symmetry groups of differential equations and, conversely, the construction of differential equations, which are manifestly invariant under some symmetry group, are important problems in mathematics and modern mathematical physics. Recently, symmetries have also been introduced in machine learning, particularly in geometric deep learning, as a way to encode properties of, e.g. neural networks and neural differential equations. To provide sufficient knowledge of the foundations of symmetries of differential equations for further study of both classical and more recent applications, this course will cover:

- Jet spaces and prolongations of point transformations
- Generating vector fields and their prolongation
- Differential invariants
- Symmetries of differential equations
- Infinitesimal symmetry condition
- Integration of ordinary differential equations
- Construction of invariant differential equations

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Form of instruction:
The teaching consists of lectures and self-study of the primary course book. The primary reading material for the course is chapters 4-6 in [1], supplemented by the background material on differential geometry and Lie groups in chapters 1-2 in [1] as needed. The book [2] provides a complementary presentation of the material focusing on the integration of ODEs based on their symmetries. The standard reference for the field is [3].

Examination:
The examination consists of hand-in problems, to which complete written solutions are to be submitted to the course coordinator.

Literature:
The primary course literature will be
[1] Olver, P., *Equivalence, Invariants and Symmetry*, Cambridge University Press, 2008

Optional reading is
[2] Hydon, P., *Symmetry Methods for Differential Equations*, Cambridge University Press, 2000
[3] Olver, P., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1998.