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A model for inclusion of all students in mathematical problem solving

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A model for inclusion of all students in mathematical problem solving

Introduction

This text will describe a model of instruction that teachers can use in their teaching to support their students to get started and tackle mathematical problems. The model has been developed in a collaborative project between teachers and researchers. The teaching model is focused on whole class instruction, but in addition to whole-class discussion, also includes individual and pair work.

The first parts of the text give the reader an insight into the development of the model and what problem solving in mathematics entails. This is followed by a description of the teaching model (pages 5–21), first more theoretically and then with a focus on its practical application. The theoretical parts are important to understand how the model works.

Developing the model

Initially, the participating teachers were inspired by a desire to find a model to involve all students in the class in mathematical problem solving — in a similar way that the teachers in first language subject (i.e. Swedish) had worked with book discussions to involve all students in reading experiences.

The need for such a model arose from the teachers' experience that some of their students recurrently ended up sitting with a blank sheet of paper in front of them during problem solving lessons in mathematics. For various reasons, the students did not want to or could not take on the task. The teachers experienced that the students lacked effective strategies for independently start working at these lessons and they wanted to find ways to teach general strategies to use at the very start of tackling mathematical problems. We believe that other mathematics teachers also share this experience and may therefore be interested in learning about our model.

An important starting point was to develop a model that teachers can use in whole-class teaching. Working with students individually is far too time-consuming and often practically impossible for the teacher.

The goal of the project was formulated as follows: To develop a model and structure for teaching that will help students get started with problem solving — to be able to reason and develop the interpretation and exploration of the mathematics in the task. This should be understood as: the goal of the teaching and learning is limited to that all students should be able to get started and tackle mathematical problems and that the goal of solving the problem lies beyond this goal.

In this way, we have focused on phases one and two in the process of problem solving in mathematics (see, Lithner et al., 2025) — *Interpretation* and *Exploration* — and we have deliberately omitted the two later phases, *Creating a solution idea* and *Implementing a solution idea*. Often, students skip exploration to instead directly search for a solution method, with insufficient insight into the mathematical properties on which the student's reasoning should be based. Therefore, it is important that students learn to explore the problem, especially in challenging problems or if the student has not yet developed functional approaches and strategies for problem solving.

While Interpretation involves understanding the information, exploration is about getting a better sense of what the problem is about and what mathematical concepts, properties and relationships — that then potentially can be used in the next phase to create a solution idea.

As a first step in developing the model, we investigated what prevented students from taking on a problem. In the subsequent discussion, we were able to conclude that there are many different types of reasons, but more specifically linked to mathematical problem solving, we found the following difficulties:

1. to understand individual words/concepts,
2. to be able to interpret and understand the context, and
3. to understand the mathematics within the problem.

Based on this investigation, we were able to identify and formulate what we called the "characteristics of interpretation and exploration", that is, what characterizes what the student needs to do to be able to interpret and explore within the mathematical problem-solving process (The process will be described later on).

Two characteristics were defined, which involve: to find the essential words of the text that support to interpret and understand the context (Helping Words), and then to construct representations of the mathematical problem (Representation). Our idea was that the formulation of the characteristics would then guide our design of instructional practices in the model (Helping Words and Representation). At this point, the language-supportive aspect of the model became clear, since students need to define and understand words and concepts. According to the model, this is done together as a whole class, which allows students to get help from the teacher and each other to extend their understanding of concepts and the mathematical context.

The project meetings were thereafter used to first plan instructional practices with a focus on one of the characteristics. The teachers then tested the instructional practices in their class, and documented their classroom experiments. At the next meeting, we evaluated and discussed the experiments and a new cycle of planning-testing-evaluation was initiated. During the course of work, we produced this text, which describes the experiences and insights generated during the work — formulated as a proposal for a model of instruction to support students in getting started and tackling mathematical problems.

In this text we share systematic and practice based tested experiences. The model has not yet been systematically tested from a research perspective. In the text the concept of "task" is used as an overarching term that includes all types of tasks the student is expected to work with. There are two types of tasks: Problems where the student needs to construct a solution and routine tasks where the student has an available solution method. In order to anchor our experiences in previous research, we in the next chapter describe the definition and problem-solving process that we used at start.

Mathematical problems och phases in problem solving

Some of the participants in this collaborative project had previously been involved in a larger research project focused on how to support students' own mathematical reasoning in mathematical problem solving (See, Lithner et al., 2025). In that research project, the following description of mathematical problems and phases in the problem-solving process were used.

Mathematical problems

A *problem* is defined as a task where the student does not have a known or in advance given solution method (see also the commentary material on the mathematics curriculum by Swedish National Agency for Education). This means that the student needs to *reason* his or her way to a solution to the problem. Mathematical reasoning means constructing a chain of thoughts that is supported by mathematical arguments. Tasks found in text books are usually *routine tasks*, that is, tasks where the solution method is provided. Teachers therefore often need to look for mathematical problems elsewhere. However, it is important to keep in mind that when students work with mathematical problems, there is still a risk that the teacher provides the solution method and thereby removes the need for reasoning to solve the problem (that then becomes transformed into a routine task).

The phases in problem solving

The process of problem solving in mathematics can be divided into four phases: Interpret, Explore, Create a solution idea, and Implement the solution idea (see also, Lithner et al., 2025). These phases are a simplification of reality, but they can help us understand what the problem-solving process means for our students and how instruction of problem solving can be designed. The first two phases are described below in more detail because the model of instruction presented in this text is focused towards these phases.

To interpret the problem

In the first phase, the student interprets the information in the task. The written information usually includes text, sometimes combined with symbols, tables, figures, pictures, etc. The interpretation includes understanding what is being asked for and sometimes also realizing what format the answer should have, for example whether it should be short or detailed. Creating a solution to the problem is not included, although in practice this sometimes starts in parallel with the interpretation. It also does not include exploring or other reasoning to arrive at properties and relationships or drawing other conclusions that are not explicitly formulated in the task, for instance, who is the oldest of the people in the problem below.

The problem below use only text and no pictures or additional symbols. If interpreted correctly, the students will understand that they should calculate Thea's age.

> *Julie, Hilde, and Thea are together 43 years old.*
> *Hilde is three years older than Julie and four years younger than Thea.*
> *How old is Thea?*
>
> (From Alseth et al., 2017, authors' translation)

A challenge for students in the interpretation phase may be to understand the meaning of a mathematical or an everyday word or phrase. In our model, the teacher helps students develop strategies to ensure they are willing and able to take on this challenge. Students gain an understanding of what it means to read and interpret a text specific to a mathematical problem and why it is important to "not hurry through", but rather to actually spend time on this phase of problem solving.

In the problem above, the student may get stuck not being able to distinguish how the people's ages relate to each other, but also in not interpreting the concepts "older than" and "younger than" based on how these concepts relate to each other. The problem contains repetitions of names, which means that students may experience the problem as messy and difficult to understand, and the student may lack strategies for structuring the information. Some of the difficulties in interpreting the problem may be general linguistic difficulties or linked to the student's reading comprehension, while other difficulties in interpreting the task are of a mathematical nature. In our model, teachers have tried to distinguish these difficulties in order to know what students need help with.

To explore the problem

After understanding the information in the problem and what is asked for, the student can move on to the exploration phase. In this phase, the student explores the mathematics by examining the situation outlined in the problem. The student learns more about mathematical properties of, for example, concepts and relationships, and tries to draw conclusions that can be useful in solving the problem. This often includes creating useful representations from the information in the problem, for

example by drawing or using concrete materials. In the problem below, students could draw a plank or use Cuisenaire rods (i.e. a type of pedagogic manipulative material used in mathematics).

> *At a wood store you have to pay per sawing if you want a plank cut.*
> *It costs 18 crowns (Swedish currency) to have a plank cut into three pieces.*
> *How much does it cost to have it cut into twelve pieces?*
>
> (From, Undvall m fl., 2020, authors' translation)

A challenge in the exploration phase is knowing how to start. There are often different possibilities and you don't always know in advance how different strategies will work. Exploration sometimes involves trying things that later turn out to be unusable or wrong. Our model shows teaching that gives students a strategy for how *to start* exploring the problem. Students can gradually develop an understanding of what exploration means and that it is an important part of learning mathematics.

In the problem above, students could get stuck not only by not including all the information in the problem in their solutions, but also by misunderstanding the mathematical situation. For example, students can calculate how many parts they get or how much the cost will be for all these parts, when they actually should find out how much they will have to pay for each sawing. They calculate the cost based on the parts of the plank instead of the number of cuttings. The fact that students encounter this kind of difficulties does not mean that they are stuck in their exploration of the problem, but rather that the students need to work more on interpreting the problem.

Instruction focused on the characteristics of interpretation and exploration

We have so far described the pedagogical problem that underpinned the desire to develop an instructional model to engage all students in mathematical problem solving and how we then addressed this desire. We have also anchored our work in previous descriptions of problems and the problem-solving process. We will now describe the two different instructional practices in the model we have developed: *Helping Words* and *Representation.*

Helping words

The instructional practice *Helping Words* aims to ensure that students know *that* they can and *how* they can approach a mathematical problem. Students who are in the interpretation phase are helped to develop strategies for finding words connected to underlying essential concepts and the meaning of the text as a whole — words that will help them understand the mathematics in the problem. With the help of the strategies, they can increasingly independently create understanding of the information in the problem. The teaching is based on the teacher encouraging and helping students to search for and underline important words in the text, as in this example:

> *Julie, Hilde, and Thea are <u>together</u> 43 years old.*
> *Hilde is <u>three</u> years <u>older than</u> Julie and <u>four</u> years <u>younger than</u> Thea.*
> *<u>How old</u> is Thea?*

The underlined words are then elaborated in class. Suggestions for how to elaborate the words together with the class are explained later under the heading *Helping Words Instruction*. By working with *Helping Words*, students can increase their understanding of the context of the problem and what the problem is about. This applies to the linguistic understanding of the text and any other representations (for example, pictures and figures) and is about the meaning of individual words and

underlying essential concepts with or without a mathematical connection. In the example, the underlined words have a clear mathematical connection, but words without such a connection may also need to be elaborated. Students need to understand all the words in order to be able to determine whether they are connected to essential concepts or the overall understanding of the problem.

Representation

The *Representation* component aims to enable students to represent the mathematical content of the problem, for example with a picture, figure, symbol or table. Choosing and/or constructing appropriate representations is a necessary starting point for students to be able to anchor their mathematical reasoning in something that is both understandable and useful for solving the problem. In the exploration phase of the problem-solving process, students investigate the situation in the problem and explore concepts and connections. Through the *Representation* component students are specifically engaged in searching for connections, start thinking about the parts of the problem and the connections between these parts, in order to create an understanding of the mathematical content of the text.

Above all, the students are given a strategy that helps them know how to start undertaking the exploration phase in problem solving. By having already selected the words that will help them understand the mathematics in the problem, they can now set about finding appropriate representations for these words. This means that they connect the mathematical content to something that they already have some experience or understanding of. In the problem below, for example, the students' experiences from sloyd (i.e. the school subject of wood craft) can contribute and help them translate "plank", "pieces", and "sawing" into an appropriate picture or that they choose to work with Cuisenaire rods (i.e. manipulative material).

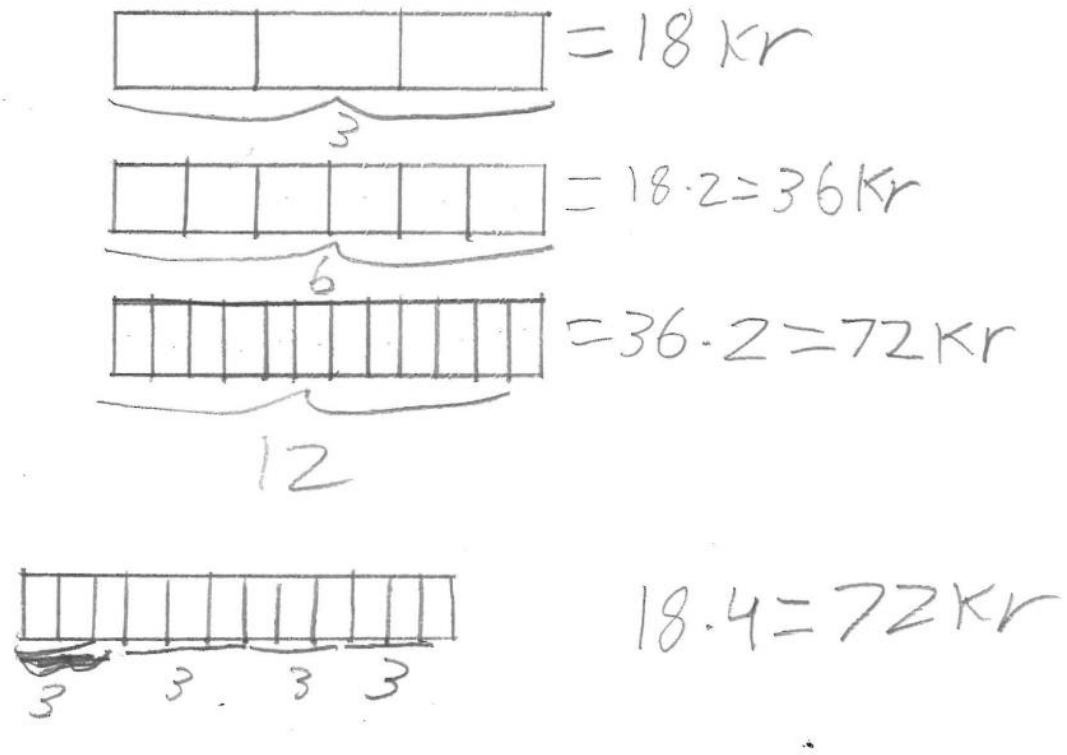
> *At a wood store you have to pay per sawing if you want a plank cut.*
> *It costs 18 crowns (Swedish currency) to have a plank cut into three pieces.*
> *How much does it cost to have it cut into twelve pieces?*

Perhaps they use a circular symbol to represent a single crown (Swedish currency), or just a dot to make it easier. Hopefully, different students have chosen different representations. Even though the focus of comparison is not between students (see below), students can share and learn about several ways to make representations to choose between. Some representations are closer to the concrete content and others are more abstract. Through the work with *Representation*, students' knowledge of different forms of representation increases and they gain experience in exploring connections between concrete and abstract content. Students can also use the same representations in different mathematical contexts, which can support understanding of seeing connections between concepts and content.

The instruction provides students with strategies and the opportunity to try different representations and reflect on and choose the form that is most appropriate in the specific context in order to better overview and understand the mathematical content of the problem. They practice to concretize, elaborate, and to do something on the paper. They learn to go between the concrete and abstract content. By communicating their representations with other students, they are offered more alternative ways of understanding the same mathematical content.

In the work with representations, it became clear to us teachers that the representation needs to be connected to the text and not to the solution to the problem. The students should represent the content of the text using pictures or symbols to make the mathematical content of the text visible. The representations are therefore not primarily part of the solution but are based on the

interpretation of the text. Another way of expressing this is that the representations should be based on what is given in the problem to form a basis for exploration that can later lead to creating a solution idea. If the student instead starts with a poorly founded solution idea and tries to represent it without having understood the characteristics of the problem, there is a great risk that they will fail. In the example with the wood store, many students represented the pieces of wood instead of the sawings. As you can see in the pictures below, this led the students in the wrong direction.



To work with the model

Now that we have introduced the two overarching components of the model, we would like to share our experiences of implementing them in the classroom. We will mix these experiences with explanations of why we chose to do things a certain way. Each teacher will always need to implement the instructional practices in a way that suits their current class and feel comfortable teaching this way. However, it is important to understand the goal of the components, so that the main purpose (the core) remains even with the adaptations the teacher chooses to make.

The two overall teaching elements build on each other. This means that the teacher first works with *Helping Words* for a period, and then adds *Representation*. The length of each period is influenced, for example, by the class's previous experiences of working in a similar way or the social climate in the class. As an example, we spent an average of three lessons introducing *Helping Words* before we introduced *Representation*. In the first lesson, the focus was on the students understanding the purpose of and familiarizing themselves with the instruction. The second and third lessons were focused on practicing together. We found it beneficial if the students also practiced the strategies on their own in parallel during the other mathematics lessons. To facilitate such training, the teacher can refer to the joint work. For example, when a student asks for help when not understanding a text-based task in the text book, the teacher can remind them of how they previously worked with *Helping Words* together.

The teachers in the project perceived benefits of working with the components at a stated lesson, i.e. at the same time(s) recurring every week. Specifically, this could mean that "on Thursday after lunch

there is always a problem-solving lesson". The students quickly became familiar with structure and norms of these lessons and what to expect. The teachers also felt that it was advantageous to, if possible, initially work in half-class setting. At the same time, there was no direct barriers to start working in a whole class setting.

The goal of the instructional practices is to get all the students in the class to take on mathematical problems. It is important to remind yourself of this goal because in other contexts, students may perceive that the goal is to solve problems and arrive at answers. It is reasonable that the students then take that perception with them. The teacher therefore needs to be extra clear and consistent in their signals to the students that in this context (during these lessons) it is not about solving the problem but instead about understanding and being able to get started and tackle the problem. For example, in cases where the teacher, after having instructed the class, needs to provide additional support to an individual student, it is important that the teacher consistently follows the same structure: Start with the *Helping words* and then support the *Representation* of the problem. Example: When the students were working individually, a student said "I can't/I don't understand" whereupon the teacher asked if he had underlined the helping words in the text. The student answered "no, but I can do it" and then worked independently and even solved the entire problem. Knowing that I, as a student, do not have to solve the problem and deliver an answer proved to be of great importance to the students. This knowledge resulted in that the students chose to engage in problem solving to a greater extent.

The work also includes planning and structuring the lessons. The teachers in the project chose to work according to the structure: Individual – Pair – All (IPA). The lessons have been 30—50 minutes long. The teachers tried to distribute the time for individual work, pair work and whole class discussion in slightly different ways depending on the needs of the class and how much time was available. In some cases, the students were allowed to think individually and then work directly as a whole class without being divided into pairs. An important insight was to dare to break the initial work in time (even though not all students felt ready) in favour of the final step, the whole class discussion.

Language and socially supportive instruction

The last overall experience that we would like to share concerns the language-strengthening function of the model. The work was experienced as particularly helpful for students with Swedish as their second language and students with other language related difficulties (e.g. language disorder), that is students with language vulnerability. The teachers experienced the work as language-supportive both directly and indirectly by being socially supportive. Short elements of pair work with *different* peers support students social competences without being too demanding. After working for one year developing the model, all teachers felt that the social climate in the class had improved. More students contributed to classroom discussions, even outside of problem-solving lessons, and students were able to share concrete tips in a pleasant and constructive way.

For the teacher, it can be important to distinguish between general language difficulties and mathematics-specific language difficulties. For example, in cases where a mathematical problem is presented in text, that text is often information-dense with many meaningful words. The reader (i.e. the student) needs to understand the information about the mathematical properties and definitions of the different concepts in the text. For example, the student needs to know what characterizes a square, unlike when the same concept is used in a literary text where the specific mathematical properties are not as important. In this way, one can distinguish between every day and strict understanding of a mathematical word.

Working with *Helping Words* becomes a natural entrance for the teacher to work further with the words in a way that strengthens the children linguistically. The question "What word have you underlined?" becomes a natural entrance into a discussion with the student, as does "Is there any word that you don't understand?" In many classes, there are more students today than before who struggle with the language, and working with *Helping Words* can make students start doing something, they get entrance into the task and can eventually move on. The teacher also gets an entrance into supporting the student. Students with Swedish as their second language are often used to ask about specific words (e.g. what "concert" means), but they often miss the meaningful small words (e.g. "than" in "older than"). These small words are harder for the teacher to predict and realize that the students do not understand and miss. To notice, questions such as "how did you think?" or "why did you do that?" can be perceived as threatening. The teacher can reformulate the sentences to "Tell me about your thinking?" or "Can you show why you did like that?" to invite the student to share their thoughts about the text to reduce the risk of students feeling they have to defend their choices.

The students' understanding of the meaning of the words is tested in the following teaching element *Representation*. When the students are told to represent the helping words, they will discover the words that they actually did not understand and try to understand them. If you in the example below do not understand the word "previous", it will be impossible to represent the problem.

> *Skol IF (i.e. the local sports club) organizes a relay race. Each team consists of five students. Each student runs 200 m further than the previous one.*
> *The race is 5 km in total.*
> *How far does the fifth student run?*
> *Answer in meters.*

We now turn to the implementation of each model component separately. We begin by reviewing the purpose of the instructional practice, as it is important to keep this understanding in mind when making adjustments that each teacher needs to make. We then present the preparatory work, the implementation of the lessons, and the further development of the component. This is followed by comments, specific experiences from the classroom, and each section concludes with a summary in which our experiences are linked back to the purpose of the instructional practice. Finally, we share the positive consequences experienced in the classroom, as well as challenges and suggestions for overcoming them.

The instruction of Helping words

The purpose of *Helping Words* is for students to know *that* they can and *how* they can get started and tackle a mathematical problem. Through the work with *Helping Words*, students are offered a strategy that helps them take a first step in starting to interpret the task. They can always start with this strategy to start working with the problem. The first step in the strategy is to find and underline the words and phrases that are essential to the problem, i.e. words, concepts and phrases that are central to understanding the problem.

> ***Problem 1***
> *Julie, Hilde, and Thea are <u>together</u> 43 years old.*
> *Hilde is <u>three</u> years <u>older than</u> Julie and <u>four</u> years <u>younger than</u> Thea.*
> *<u>How old</u> is Thea?*
>
> (Från Alseth m fl., 2017)

In this example, there are many words with a mathematical connection that are central and that the problem solver needs to understand. The word “together” shows that the girls’ ages have been combined. The concepts “older” and “younger” are everyday words in one sense, but they also have a meaning in a mathematical sense. The question is always central in a problem and in this example the student needs to understand that it is the number of years of one of the girls that constitutes the answer, not, for example, who is the oldest. Finally, all the numbers that are reported appear to be important, which is not the case in all mathematical problems.

By working with *Helping Words,* students can learn that not all words in the text are central to the problem. For example, some words are included for grammatical reasons. Sometimes the opening sentence is not at all important for the mathematical context, while words in subsequent sentences are of great importance for the student to understand the mathematical situation.

In the *Helping Words* strategy, students work with one sentence at a time, either individually or in pairs. By processing the text sentence by sentence, students slow down and pay attention to each word in the sentence to determine which of them are essential. This is followed by the second step in the strategy, which involves the teacher discussing with the whole class which words the students have identified and then discussing the meanings and role of the words in the context. The goal is for the student to gain insight into which words create the mathematical situation. Differences in the students’ ideas and thoughts can contribute to new insights, and the teacher can consciously direct the students’ attention to how the *Helping Words* and understanding these words help the students to understand the problem as a whole. The goal is for students to leave the feeling that "I don't understand anything", because that feeling can result in "there’s no point in trying".

The following description will show how teachers can work to help students develop this strategy. We will first discuss how to introduce *Helping Words* in a class without previous experiences of working in this way, and then how to further develop the work.

Preparatory work

To prepare the work the teacher should:

- Choose a problem adapted to the group of students,
- create opportunity to show the problem via, for example, a projector for whole class discussion,
- print a copy of the problem to each student, and
- prepare themselves on which words are essential and how they can be explained.

Lesson implementation

The lesson is implemented as follows:

1. The teacher describes the purpose of the lesson and explains that all words in a text do not have the same supporting function for understanding the problem. During the lesson the class will identify and together describe which words are supporting for understanding the text. The goal is to interpret and understand what the problem is about, but not to solve the problem. If the students want, they can solve the problem later, but the purpose of the lesson is not to find a solution method.
2. The teacher shows and/or reads the problem aloud to the class. The students also have a copy printed on their own paper so that they can later underline words and make their own notes.

3. Together with the class, the teacher works on the meaning of the words, sentence by sentence. Start with the first sentence and let the students find the helping words. Highlight the words and discuss them together in class. Then read the next sentence and follow the same pattern. If the teacher wants, the students can work individually or in pairs to identify the words and then highlight the meaning of the words in whole class discussions.

Examples of problems to work with

In these examples, the words that may be important for understanding the problem have been underlined.

> ***Problem 2***
> *Sara is standing in line to buy tickets for a concert.*
> *She is fourth in line from the front and fifth from the back.*
> *How many people are standing in line?*
>
> (From Undvall m fl., 2020, authors' translation)

> ***Problem 3***
> *The rectangle is twice as long as it is wide.*
> *The perimeter is 12 cm.*
> *What are the length and width?*

In the work on Problem 2 above, one student chose to underline the word "concert". In the whole class discussion, the teacher asked the question: "If we had worked on the same problem and it had said circus instead of concert – Would it have made any difference?" The class concludes that the answer is No and the teacher explains that the word does not need to be underlined, since it is not crusial to understand the problem. The students gain experience in determining whether a word is essential or not. They can also understand why it is unnecessary to underline a word that they are not going to work on further.

Note that these are both mathematical terms and words that do not have a mathematical connection in themselves but are important for understanding the mathematics in the problem. The word "in line" is an example of a helping word of the latter type in Problem 2. The words "place" (only applicable in Swedish wording), "from the front" and "from the back" cannot be understood without first understanding the word "in line" and these three words are in turn crucial for understanding the problem. "Fourth" and "fifth" are examples of mathematical words that the student needs to understand in order to fully understand the problem. The question in the problem is often crucial. In this case, "How many" is particularly informative in the question for understanding the problem.

Being able to decide which numbers are relevant and which are not in a problem is important, but often not so easy to decide. In Problem 1, all the numbers (43, three, four) are relevant, but this is not always the case. In one of the classroom discussions, the teacher stopped the students and asked "But what does older than actually mean?" because several students were stuck on the word "than". The students had to try to summarize the essence of the words, after which one student described how "than" has to do with comparison, e.g. when comparing and showing differences between different numbers. The other students thought they received a good explanation and the situation was advantageous both because students are "forced" to express themselves and because students can understand each other more easily when they use wordings that suits their peers.

Choosing problems appropriate for *Helping Words*

Choosing this type of problem is based on finding a text with a context known to the students. In that context one or more mathematical concepts are included, for example, concepts of comparison. These (and possibly other words) are underlined, discussed, and called "helping words".

Further examples:

Problem 4

Fabian was 62 years old when Fredrik was 36 years old.
How many years ago was Fabian
exactly three times as old as Fredrik?
(Answer: 23 years)

(From Alseth et al., 2017, authors' translation)

Problem 5

There are 24 students in class 3b. On a school activity day, a third of the students choose to go bowling. A quarter choose to play tennis.
The rest of the students choose to go for indoor climbing.
How many students choose to go climbing?
What proportion of the students choose to go climbing?
(Answer: 10 students climb, the proportion is 5/12)

Further development of Helping words

Gradually, students learn that words sometimes need to be put together to create meaning, for example "more than". That is, the *helping words* do not have isolated meanings but need to be understood from the context of the task or sentence, and, above all, words *can* have different meanings in different contexts within problem solving in mathematics. Compare the examples below:

Problem 6a

At Coop (i.e. a Swedish grocery store), a soft drink costs 17 crowns (Swedish currency).
At Ica (another grocery store brand), a similar soft drink costs two crowns less.
How much does a soft drink cost at Ica?

Problem 6b

At Coop, a soft drink costs 17 crowns.
It costs two crowns less than a similar one at Ica.
How much does a soft drink cost at Ica?

Having students compare two similar problems with each other is a way to make students aware of the context and the meaning of the individual words. We call this a further development of *Helping Words.*

Preparatory work

To prepare the work the teacher should:

- choose two problems that can be compared and that are adapted to the group of students
- create the possibility to show the problem via, for example, a projector, that is for whole class discussion,
- print a copy of the problem for each student, and
- prepare themselves on which words are essential and how they can be explained.

Lesson implementation

The lesson is implemented as follows:

First follow the same procedure as before (points 1–3). Work with task 6b in the same way and then compare task 6a with task 6b (points 4–5). The comparison is central to the purpose of the lesson – to make it clear that concepts are in a context and therefore do not have isolated meanings. The students need to be able to clarify that words can change in meaning based on grammar and context.

1. The teacher describes the purpose of the lesson and explains that not all words in a text have the same supporting function for understanding and that words gain meaning from the context in which they appear. During the lesson the class will identify and together describe which words are supporting for understanding the text by comparing two problems. The goal is for the class to interpret and understand what the problems are about, but not to solve the problems. If the students want, they can solve the problems later, but the purpose of the lesson is not to find a solution method.
2. The teacher shows and/or reads the first problem aloud to the class, the students also have a copy printed on their own papers so that they can underline words and make their own notes.
3. Together with the class, the teacher works on the meaning of the words, sentence by sentence. Start with the first sentence and let the students find the helping words and - if the teacher wants – which words are "unnecessary". Mark the words and discuss them together in class. Then read the next sentence and then follow the same pattern as before. If the teacher wishes, students can work individually or in pairs to identify the words and then highlight the meaning of the words in whole-class discussions.
4. Work on the next problem in the same way.
5. Discuss in whole class how the problems differ.

Comments on the implementation

In the classroom discussion, the discussion and reasoning itself are central. Here, the students are required to formulate what similarities and differences the problems have and what this depends on in terms of textual formulations. The students should experience that they can think freely, but the purpose of the work is still for the students to practice identifying underlying concepts and to practice understanding what the similarities and differences in the two mathematical situations depend on.

The question, which often comes last in the problem, does not need to be prioritized as long as the students are working on interpreting the rest of the text in the problem. This is because they may be tempted to focus on the solution idea instead of interpreting all the text related to the problem. If the students find it difficult to stick to the interpretation of the problem and would rather focus on solving the problem, the teacher can delete the question and present it after the students have interpreted all the other sentences.

Choosing problems appropriate for further development of Helping words

Choosing this type of problem is based on that similar formulations of the problems create different situations. In other words, the same concept can have different meanings in different formulations. It is this difference that is highlighted in the further development of Helping words. Further examples:

Problem 7a
Per and his three friends are to share 36 candies equally.
How many candies does each person get?
(Answer: 9 candies.)

Problem 7b
Per's three friends are to share 36 candies equally.
How many candies does each person get?
(Answer: 12 candies.)

Classroom experiences

In the following text we will discuss experiences from introducing *Helping Words* and the further development by working with comparative problems.

Firstly, it may be beneficial to not hand out individual papers to the students in the beginning of working with *Helping words.* It may then be easier to gather the students for a joint discussion, especially considering the students who work quickly. If the students at this early stage have individual papers, it is difficult to get the students to just underline words and then stop, because (based on their previous experiences) they often focus on solving the task. When students have solved the problem, it may be that they do not engage in the discussion because they feel that they are "done". However, once the students have learned about *Helping words*, we did not see any difficulties with the students taking notes on their own papers.

When the students have gradually understood the purpose of the lesson – that they should only interpret the task – it is advantageous for the teacher to hand out individual papers to the students. For the student it becomes easier to compare and perceive differences between how their own work/thinking and other students' work/thinking when supported by their own notes. From the teacher's perspective, not handling paper can save time, but at the same time it becomes more difficult to assess and get an overview of possible student misunderstandings.

Second, we want all students, regardless of how far they have come in their learning, to engage in the joint discussion to create a joint learning opportunity. Our experience is that students are not always aware that what they say is helpful to other students when they are, for example, answering a question. Therefore, we believe that it was very important to convey the purpose of the lesson to the students: that the discussion is central to the learning process.

We also agreed on that the discussion about which words are essential and which are not essential for understanding the context is so important that the discussion is also needed in classes where

Helping Words have been used for a longer period of time. Students who have progressed further in their learning are given the opportunity to summarize the meaning and role of the *helping words.* They can then consolidate their knowledge, while students who need further learning are given the opportunity to repeat the content.

When the teachers tried the comparative problems in the classroom, several of the students initially thought that Problem a and Problem b were the same. The students did not see the small differences between the tasks that ultimately had a big impact on the interpretation. When the two problems were compared with each other, the teachers were able to help create a discussion between the students about how easy it is to make mistakes if you do not read carefully and pay attention to all the words. For the students, it was a useful experience to discover that “I actually made a mistake even though it was such an easy task” or “17–2 is easy, but I made a mistake anyway”.

In the discussion with the students and the comparison between the problems, the teacher can ask reflection questions: "[Student 1] has arrived at .... and [Student 2] has arrived at .... (which is something different from what [Student 1] arrived at). How do you think [Student 1] and [Student 2] were thinking?", and/or "Which words do you think [Student 1] and [Student 2] had underlined?" The teacher’s role is thus to help make the students’ thinking visible so that they can broaden and deepen their mathematical reasoning linked to the meaning and role of the words in the context.

When working with students, it is important to remember that the aim is for students to have strategies for getting started and tackle the problem. The aim is for students to learn to identify important/relevant words and concepts that they then further elaborate on. This means being able to evaluate words, which also means being able to sorting out words and concepts that are less important/relevant for understanding the problem. This purpose and aim means that it is not decisive if students choose the “wrong words”, especially not at first. This, on the other hand, does not mean that what words the student chooses is unimportant, but the student needs both encouragement and guidance to successively reach the aim of choosing “right words”. The important thing is that the student finds enough relevant helping words to get started with a proper interpretation. The teacher needs to balance these needs and adapt their feedback to different students. Pair work and whole class discussions also serve as feedback to the students, i.e. students can experience both confirmation and challenge when they compare their own thoughts with other students’ thinking. However, in the beginning and for unconfident students, it is important that students receive sufficient guidance so that they understand what is meant by the strategy offered in *Helping Words*.

Third, as we mentioned before, we found that it was much more common for students to choose “unnecessarily many” words than to miss words. Our experience is that students learn from the teacher and from each other and realize the advantage of only selecting and marking the “necessary” words, but sorting out words is likely more difficult than choosing words. A little more time and guidance may therefore be needed for taking this step to sort out “unnecessary” words. Moreover, the discussions can also lead to students starting to recognize different types of problems, which allows students to bring experiences to new problems.

Lastly, one of the teachers had the opportunity to introduce *Helping Words* in parallel to a new class and further develop its use in another class. The teacher noticed a difference between these two groups. Students who had previously worked with *Helping Words* chose to put words together to a greater extent, e.g., “double as long”, “how many”, “divide equally”, and this was without the teacher talking about this. The fact is that the meaning often becomes clearer when you put the words

together to form a whole. It may be easier for students to start with individual words and then learn to combine words that together form a whole with a specific meaning.

Summary

Going back to the goal of *Helping Words*: Students should get started and be able to take on working with mathematical problems. They should know that they have strategies for how they can start. A first step might be to look at a word, then look at the other words and take further steps towards understanding the whole and the context. Some students need to be given the opportunity to make such a gradual development. They need to be given an introduction to start with something at all, and then gradually move on, and then you as a teacher can think that it is okay and necessary for them to start with the individual words.

If you imagine a general progression from only seeing individual words, to seeing parts that are connected, and to seeing the whole context, you should bear in mind that students have come to different stages in this process. Some students need guidance in finding words in the text while other students can already put the words in a mathematical context. The teachers did not experience that the fact that students had come to different stages in their mastery to interpret the problem was a big deal in the whole-class setting. Regardless of where the students are in their learning process, they have the opportunity to develop their mathematical competences through the discussions, either through repetition or deepening their knowledge.

In the introduction of *Helping Words*, it is pointed out that the teacher should *not* signal that the problem should be solved. In the further development of *Helping Words*, with students who have become used to this practice, we chose to deviate from this principle. The solution to the problem is still not in focus, but the differences between the problems were clearly visible when the calculated answers were different. The students could then compare the *helping words* of the problems together with their respective solutions.

Helping words – Opportunities and positive consequences

In line with the aim of the model, we experienced that the threshold for entering the problem-solving process seems to be lowered through the teaching element *Helping Words*. A common view was that students – regardless of their level of knowledge – can participate in the group discussions, given that the focus is not on solving the problem.

The work with *Helping Words* is language supportive. We have previously described the benefits for both the students and the teacher – who gets a natural entry to provide support. The work can be experienced additional challenging when it comes to students with different types of linguistic vulnerabilities and problems with slightly longer text. In such cases, it is important to remember that the purpose of the instruction described in this text is to create opportunities for all students to participate. Students with linguistic vulnerabilities often end up with learning difficulties – the purpose here is to make it easier for them in a way that works in a whole-class context.

Helping words allows all students to be involved and it was experienced most favourably for those students who do not think they “get” the mathematics. Students who are unsure of their own capability could be involved in identifying the *helping words* and included in the discussions about them. In this way, all students can feel competent and participating, and the teaching becomes inclusive.

Helping Words can also support students get involved in the work so that their motivation increases. They begin to believe in themselves, see the value of what they are doing and it feels more fun.

Student activity increases and everyone can participate at an early stage of the lesson. We experienced a high level of commitment from the students. They wanted to find more and more words and began to reflect on and use mathematical words when they talked and more students were also motivated to take on and later solve the task. The students appreciated and found it fun when everyone wanted to contribute with suggestions and explanations. They showed that they wanted to involve each other and that their peers should understand.

Also, we noticed that more and more students began to remember to start with the (sub)strategies they have learned through the *Helping Words* work when they got stuck. The strategies make students stop and think, “What is important in the text?”, and the students’ opportunities to tackle the problem increase. Thus, the instruction strengthens students’ ability to reflect on their interpretation attempts, which is a crucial ability for becoming a successful problem solver in mathematics.

Another advantage of *Helping Words* is that the teacher can refer back to the joint work in the event that an individual student needs additional support to tackle a problem. Although the students’ participation and commitment are facilitated, this does not mean that all students immediately want and can take on the problem. Individual students need different lengths of time and paths. The teachers told that they had begun encourage and support the individual student in a different way compared to before the project by reminding the student what they did in the whole class work. For example, when a student says "I don't understand anything", the teacher can refer back to how they worked in the class and encourage the student to do the same thing themselves. Several teachers said that in such situations they previously often started to guide the student in solving the problem. They did not support the student to take on the task on their own. Now, the student keeps the initiative and has a strategy for and training in getting started and tackle mathematical problems independently. For some students this comes naturally, while other students need the recurring practice to reach automaticity and routine. *Helping Words* makes the strategies accessible to all students and their use becomes part of mathematics.

We have experienced that the students became increasingly used to using the strategies and increasingly began to independently produce the *helping words* on their own. We believe that the consistent and recurring use of *Helping Words* is the reason for this development. There are even examples of students who reminded the teacher that they had to go through the *helping words* together as part of the problem solving process, which shows that the work was perceived as meaningful from the student’s perspective.

Helping words – Challenges and Solving them

As a fact, in *Helping Words*, it is advantageous if students dare to ask if they do not understand what a word means. However, asking a question reveals that you do not understand, which can make you feel stupid and bad. It is therefore far from obvious that all students dare to ask. Especially in the beginning, before students have experienced and understood the benefits of *Helping Words*, the teacher needs to have strategies for managing social interaction in the class.

One example of such a strategy is that the teacher can show what a student who asked a question has actually managed to do and what they need to know. If a student, for example, even though it feels embarrassing asks what a rectangle is, the teacher can highlight that the student managed to determine what was a key word and that it was important to understand this word. The teacher can then discuss what a rectangle is.

Moreover, the teacher also needs strategies for dealing with situations where students question their peers’ explanations of words and concepts. A student who questions a peer’s explanation shows

engagement and active thinking about the meaning and role of the concepts, something we want them to do. At the same time, students need to learn and practice appropriate ways to ask questions and give each other feedback so that they do not discourage each other. When a student says "...then my thinking was wrong," the teacher can turn this into something positive, and show that the class got an opportunity to learn something new.

In the whole class discussion, there is a balance between acknowledging all students' contributions and showing differences in quality and relevance. It is important that students feel that their contribution is significant. Having made the effort to think should be experienced as a good enough contribution. More generally, it is valuable if students realize that sometimes making mistakes is part of mathematical problem solving. The only way to avoid mistakes is to always know in advance what to do, but then it is not problem solving. Making mistakes – and learn something from them – is better than doing nothing at all because you are uncertain.

A general challenge with *Helping Words* is that it takes time. Teachers often feel that there is a lot that needs to be done during the mathematics lessons that are available during an academic year. The teacher therefore needs to believe that the work of interpreting and understanding problems will pay off in the long run. Since whole class discussions are central to the students' learning, it is important that there is time for these discussions. If the lesson time is too short, our experience is that it can be better to postpone the discussion to the next lesson. The same applies if you feel that the class has completely lost focus. In that case, it may be better to cancel the work and instead continue later on.

Finally, a pitfall in *Helping Words* is a confound with other instructional approaches saying that the students should search for "signal words" (other names also exist) which leads to the meanings of the words being linked to a particular solving or calculation method. See for example tasks 6a and 6b above that when using a superficial interpretation of signal words leads to using the same solution method – which is wrong. *Helping Words* does *not* mean to work with isolated meanings where the students are asked to identify words that help them choose a method to solve the task. Interpreting the words in such way lacks mathematical meaning and provides superficial clues that are effective and helpful in the short term, but which in the long term can be harmful to the students' development of mathematical understanding. *Helping Words* is not aiming at helping the students choose a solution method, but at providing the students with strategies for understanding the mathematical situation and the context of the problem.

As an example, "in total" does not necessarily mean that something should be added, but it can be a way to understand the mathematical situation. In comparison to reading in other contexts, identifying signal words would be considered "search reading", while identifying *helping words* promotes students making inferences (reading between the lines and drawing conclusions). For example, we want students to understand that there is a relationship between two objects when they encounter "less than" and then think about what the relationship is. We do not want them to think "Then I should use subtraction."

In *Helping Words* students gain insight into the problem and are given the opportunity to create an internal picture of it. This helps them to be able to choose a representation for the mathematical situation in the next step.

The instruction of Representation

Helping Words provides strategies to the students for the first step of taking on and understanding the problem. The instructional practice *Representation* provides strategies for taking on the next step. This step involves an in-depth interpretation and exploration of the mathematical ideas that the

problem is about. The aim is to master translating the content of the problem into useful representations, or in other words, to find representations of concepts and/or relationships. The idea of *Representation* is that students should know that they can start representing the problem by finding representations for the *helping words* they identified in the text. The continued joint work allows students to get ideas about possible alternative representations and they can compare and evaluate representations that might work better. In the example of sawing the plank, one possible representation was to create a picture, another was to use Cuisenaire rods.

> ***Problem 8***
> *At a wood store you have to pay per sawing if you want a plank cut.*
> *It costs 18 crowns (Swedish currency) to have a plank cut into three pieces.*
> *How much does it cost to have it cut into twelve pieces?*
>
> (From Undvall et al., 2020)

When students compare these two representations, they might conclude that a student picture is more appropriate because the plank as a whole becomes clearer in a picture compared to using three rods representing each sawn-off part. Allowing students to see each other's representations can also help broaden and deepen their own understanding of the mathematical content because the representations complement each other.

Since exploration of mathematics problems includes to not always know in advance what will or will not work, the teacher may need to help the student realize that experimentation is a natural part of problem solving and an important part of learning. Gradually, students learn to "figure out the meaning of the problem" on their own, which is something students often ask the teacher to help them with. The representation, such as a picture, helps to see and understand the parts and the whole of the problem. Words and concepts take on a mathematical meaning.

Once again we go into the concrete work in the classroom. We describe the preparatory work, the implementation of the lessons including comments, experiences from the classroom, a summary with a re-connection to the purpose, the positive consequences, as well as challenges and their solutions.

Preparatory work

To prepare the work the teacher should:

- choose a problem adapted to the group of students
- create the opportunity to show the problem via, for example, a projector for whole class discussion.
- print a copy of the problem for each student.
- bring manipulative material, and
- prepare oneself for how the mathematical content can be represented in different ways, for example by solving the problem oneself and reflecting on the solution.

Lesson implementation

The lesson is implemented as follows:

1. The teacher describes the purpose of the lesson and explains that the text in the problem can be represented using, for example, manipulative material or pictures that the students draw themselves. During the lesson the class will represent the words that are essential to

understanding the text. The goal is that the class will gain a deeper understanding of what the problem is about through their representations, but not to solve the problem. If the students want, they can solve the problem later, but the purpose of the lesson is not to find a solution method.

2. The teacher shows and/or reads the problem aloud to the class, the students also have a copy printed on their own papers so that they can underline words and make their own notes.
3. The students underline the helping words on their own copies.
4. After a short while, highlight and discuss the helping words together according to the implementation of the previous lessons with *Helping words*.
5. Ask the students to create *representations of the helping words*. They can either draw a picture or use manipulative material.
6. One student shows their representation on the board using, for example, a document camera or projector. The teacher asks if anyone has represented the word in a different way. If there are multiple representations, they are shown in the same way and the teacher helps the students find similarities and differences between the different suggestions. Eventual symbolic representations are connected to other representations in the same way.

Comments on the implementation

While *Helping Words* worked relatively straight forward, we found that this instructional practice required more planning and reflection before fully working. We discovered that how students are introduced to create a representation for the mathematical content of the problem matters. It is important that students understand that it is the *helping words* that is supposed to be represented in order to explore the mathematics in the problem. Our experience was that students otherwise tried to create a solution idea using representations, which is *not* the purpose of the instructional practice. When the teacher comes to point 5 above, the instruction to the students therefore needs to be clear: It is the *helping words that are to be represented*, nothing else.

Students can benefit from working with representations in the same way as they did with the *helping words*, that is, taking sentence by sentence to ensure that they have represented all the *helping words*. Some words will be collected in the same representation, which means that one representation per word is not required. If students understand how the *helping words* are logically connected in the same representation, teachers should encourage this. For example, a student can draw X in different circles where the X represent balls and the circles represent boxes. This representation shows that the student has understood that there will be a certain number of boxes to be filled with balls even though the student has not drawn either balls or boxes.

Choosing problems appropriate to *Representation*

Choosing this type of problem is based on that something needs to be represented to create a deeper understanding of the context and that the representation does not automatically solve the problem. Further examples:

Problem 9
Ida and Noel are going to share a pizza that weighs 360 g.
One piece should be twice as big as the other.
How many grams does the larger piece weigh?
(Answer: 240g)

Problem 10
There are 10 different animals in a fenced field. There are twice as many sheep as cows. There are as many horses as sheep.
How many of each animal are there?
(Answer: Sheep = 4, Cows = 2, Horses = 4)

Problem 11
A lawn is 24 meters long and 15 meters wide. A fence should be built around the lawn. The distance between fence posts should be 3 meters.
How many posts are needed?
(Answer: 25 posts)

(From Undvall et al., 2019, Authors translation)

Problem 12
Wilma and Mika take an elevator in a high-rising building. They first go up five floors, then down seven floors and finally up nine floors.
The elevator has now stopped on the twelfth floor.
On which floor did Wilma and Mika start?
(Answer: Fifth floor)

(From Undvall et al., 2020, Authors translation)

Classroom experiences

Representation was experienced a link between the phases of interpretation and exploration, without considering solving the task. Taking sentence by sentence and drawing/making notes on the paper to "figure out" the meaning of the problem can create an understanding of its parts and the whole. The teacher may need to remind the students of this step-by-step procedure and also that it is the underlined (helping) words that the students are supposed to represent.

An unexpected and interesting experience was that students who had not progressed that far in their mathematical development (e.g. struggling students) often made clearer representations and more easily described their thinking compared to students who generally perform better in mathematics. A possible explanation is that the latter students perceive that they have a clear solution idea before they have reflected on the entire content of the problem. Maybe they made a superficial interpretation that led to errors. They may also think that making a representation is too time consuming and therefore go straight to the answer without properly exploring the problem. The

teachers perceived that these students sometimes miss important parts (words) of the text and have more difficulty describing all the steps, something that all students eventually need to be able to do. In Problem 8 (the wood store), several students immediately started to find solutions using the symbolic language of mathematics and then went on a misleading track without noticing themselves. Several failed to take into account the plank as a whole. The students focused instead on the pieces that make up the plank, not on that the plank should be cut.

Summary

Working with *Helping words*, the teachers experienced that many students were provided tools to take on problems by interpreting the text. However, they lacked strategies for exploring the mathematics symbolized by the words. Based on this, an instructional practice was created where students are allowed to represent the *help words* using, for example, pictures, tables or manipulative material.

Through whole class discussions, students learn strategies for both interpreting and exploring the mathematics of the problems. These strategies can then be used in individual work in the future. If students get stuck and want help, the teacher can remind the students of the strategies instead of helping the student understand the problem.

Representation – Opportunities and positive consequences

Also in the exploration phase, teachers found the students more relaxed and less pressured due to the focus on taking on problems, not solving the problem. The instructional practice *Representation* served as a way to “pull students into the problem”, to make them want to and start trying to explore the mathematics in the problem.

Another positive experience was that “students play with the numbers” to move forward. Even if their actions not always turn out right, the students do something, which is better than not being active at all. The experience was also that the students continued finding these lessons fun. In *Representation,* students can start at a level of concretion that suits them. One student may need to draw the dogs described in the problem to understand what is required, while another student can jump straight to representing each dog with a line.

The goal is for students to be able to "figure out the problem" on their own, which is something that students often ask the teacher to help them with. It is about being able to translate the content of the problem into a useful representation, to find representations for concepts and/or relationships. Also, the psychological effect of the work also appears to be important. Knowing that there is an easy strategy that I can use to start and that I will be able to handle, the student can overcome feelings of overwhelmingly difficult. In a group of students with neuropsychiatric difficulties, there were students who refused to take on problems because they contained too much text. The instructional practice meant that the students began to view problem solving as a step-by-step process where you always need to take a first step, namely interpreting the task. The view of problem solving as "either-or", "black-or-white" or "I can-I can't" was blurred.

When students dare to try, it can lead to learning and development of abilities. The student’s creativity is strengthened by trying new things. The student's ability to reflect is strengthened when the student themselves judge whether their exploration leads forward or not. The student’s view of mathematics and themself as problem solvers is strengthened when they experiences that their own initiatives are valuable even if everything is not correct from the start. In other words, the student trusts his own mathematical reasoning. The student’s creativity, ability to reflect, and mathematical authority develop over time. Training and shared learning strengthen the students, but it is based on the students having a strategy for how they can start exploring.

Representation was also experienced positively from the teacher's perspective. Also in this instructional practice, the teacher gets a natural entry to provide support by asking questions linked to the words in the problem and their meaning, for instance: "What does one sawing cost?". The teacher can also help the students to compare each other's representations and primarily pay attention to how the representations relate to the *helping words.* Another advantage that was highlighted is that the whole class discussion strengthens the independent work of many students, which means that the teacher has time to help individual students who need help.

One conclusion using *Representation* was that students' different levels of knowing play less role when the goal is not to solve the problem. The work is focused on involving students who previously were unwilling or unable to take on problems, but the other students also receive strategies that they can use, for example, when solving more difficult problems.

The shared classroom learning is raised to higher levels and the students show that they want to be involved and create learning opportunities for their peers. This has to do with working with mathematics together. The students become focused on telling and showing, but without focusing on solving the problem. This is positive because the students show that they want to understand the problem and the risk of students who do not dare to share is reduced.

One highlighted aspect of shared learning was comments from students expressing that "we have done this in a similar way in previous tasks". It is a strength when students discover that types of problems recur. This shows that they compare tasks on a deeper level, e.g.: "but there is always one less when you cut!". Students move from a superficial to a deeper, more reflective use of their experiences. The teachers experienced that more and more students made such comments.

Representation – Challenges and Solving them

When the teachers discussed their experiences with *Representation,* it became clear that the outcomes in the classes had been very different. In some classes, many students started by drawing a picture and then used numbers (symbols). In one class, there was a large spread in which representations the students used, while the students in another class had basically represented the mathematics in the same way. We could not really understand the reasons for the differences, but the unpredictability makes it more difficult to plan and manage teaching. As a teacher, you therefore need to be prepared to provide complementary suggestions if the students' representations are (almost) the same.

The next challenge concerns the goal of the instruction. Although the teacher consciously and clearly signals that the focus is on taking on problems, situations can arise where the students are unplanned led towards solving the problem. One example was when, before the students were to begin the individual work (**I**PA), the teacher asked the question "Do you have any questions?", whereupon a student asked "Can I use tables?" i.e. a solution idea. The consequence was that a great deal of the other students immediately started using a table, which meant that the students opted out their own in-depth interpretation and exploration. The joint comparison in pairs (I**P**A) also became less interesting because many students acted in the same way. As a teacher, you can perhaps avoid this by being more specific when checking that the students know what to do instead of asking the class such an open question.

The difference between teachers' and students' views on goals is also evident in test situations. Despite the fact that the class had worked with a focus on interpretation and exploration during the whole school year, students turned back into saying "I can't do this" in the context of test situations. The teacher may then need to provide an extra reminder about *Helping Words.* Another idea

proposed was to use this situation as an opportunity to explain that the strategies they had practiced in *Helping words* can help the student get at least some of the test scores for the task, for example, on the National Exam in mathematics. The assessment criteria can become clearer to the students.

Opportunities to learn from mistakes are not necessarily appreciated from the students' perspective. One mistake that several students made in this context was to, without reflection, use a method that had worked before. For example, they relied on proportionality in Problem 8 (The wood store) and thus made the calculation 4x18kr. The teachers perceived that the students were comparing this problem and previous problem(s) on a superficial level. The challenge is to help the students become aware of this pitfall and to realize that they need to make a more careful comparison, that is, to learn from their mistake. It is important that as a teacher choose to see this mistake as something the class can learn from together. A general misconception can be reflected on by discussing what the problem actually requires, for example by asking the students how they represented the word "sawing" in their drawings. The teacher can then point out the difference between "sawing" and "piece". Another possible way is to choose two problems where the superficial method works for one problem, but not the other.

Another challenge is linked to the whole class discussion (IP**A**). When students were asked to identify similarities and differences in representations, it was common for students to compare the concrete (Ex: he has drawn balls and I have drawn monkeys) instead of how the mathematical content was represented. To help students with this advanced task, the teacher needs to lead and help deepen the discussion. While the differences between students become an asset as students can learn from each other, the teacher needs to be prepared to handle student differences in understanding and tempo.

Conclusions

A very pleasing conclusion from this project is that the desire to work with problem solving has increased in all classes. The students feel they master problem solving at least at some degree, which makes it easier for them to continue their work with the problem. The students are also trained to dare to discuss together, which has made the climate in the class more permissive and positive. The language-strengthening aspect of the model became increasingly prominent during the course of the project. We noticed that many students who struggle with language issues finally got strategies to be able to work independently in the initial work. These students will still need help from the teacher, but not all students need help at the same time when most students can start working with the problem independently.

Since we as teachers see that many of the students use the strategies when they work individually, we conclude that they feel that they are supported by them. Instead of helping the student with the solution method of the problem, the teacher can remind the students about the strategies. This allows the student to continue working on their own with their own reasoning instead of receiving guidance and clues from the teacher. This strengthens the student's mathematical authority and self-confidence in the long term.

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